THE UNIVERSITY OF ALBERTA

SPIN AND GRAVITATION

BY

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To my mother.

ABSTRACT

This thesis is concerned with several aspects of the role played by spin in general relativity, more specifically, in particle equations of motion and in field equations for spinning matter. These equations are derived from variational principles, where the Lagrangian has some dependence on terms associated with spin, in addition to the usual dependence on the metric, external fields, etc.

Chapter One presents a general discussion of the thesis and introduces the notation used. A more detailed introduction to notation appears in Appendices One and Two, which discuss invariance identities satisfied by a scalar Lagrangian, these having important use in Chapters Two and Three.

In Chapter Two, covariant equations of motion for a single spinning particle are derived, and the special case of motion in an external electromagnetic field is discussed.

Einstein's gravitational equations for a medium with intrinsic spin are examined in Chapter Three. The form of the total energy momentum tensor for such a medium is found and it is shown how the equations of Chapter Two may also be recovered from the four dimensional action integral.

Chapter Four generalises the work of Chapter Three, an action integral depending on any number of derivatives of the fields is considered, and finally, the derivation of particle equations of motion from the Dirac equation is discussed in Chapter Five.

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TABLE OF CONTENTS

CHAPTER ONE

GENERAL INTRODUCTION AND NOTATION

The main concern of this thesis is an examination, in the general relativistic framework, of the field laws governing fields generated by matter, where the matter is assumed to possess intrinsic spin. By field laws we mean the gravitational field equations and equations for any other fields present, such as the electromagnetic field.

For a medium with no internal spin the Einstein equations $G^{\mu\nu} = T^{\mu\nu}$ are the usually adopted gravitational equations, coupling the Einstein tensor to the total, symmetric, conserved (i.e. $T^{\mu\nu}{}_{|\nu} = 0$), energy momentum tensor $T^{\mu\nu}$ of the medium. In the spinning case one may either keep the Einstein equations and examine the constituent parts of the energy momentum tensor, to find what terms attributable to material spin appear in $T^{\mu\nu}$, or one can try to modify or generalize the Einstein equations themselves.

The latter approach has led to several new theories [1]. There is the generalisation of Einstein's theory, originally due to Elie Cartan, which takes space time to be a four dimensional differentiable manifold with a metric tensor and affine connection compatible with the metric, but with the connection no longer taken to be symmetric. The antisymmetric part of the connection is then related to the density of intrinsic spin. Another idea is the suggestion of Sciama [2], that the gravitational field produced by particles with spin should be described by a non-symmetric

metric tensor $g^{\mu\nu}$, the energy momentum tensor thus, in general, being also non-symmetric.

In contrast to the above, the approach taken in this thesis is to retain the orthodox theory of a Riemannian space time with a symmetric metric tensor and symmetric connection. We find the form of the total, symmetric, conserved, energy momentum tensor, a form containing terms due to intrinsic spin and one that reduces to the usual energy momentum tensor in the absence of spin. Variational principles are adopted, the contribution from internal spin stemming from the dependence of the Lagrangian upon a tetrad field $e_{\mu}^{(a)}(x)$ describing particle spin.

Using a statistical approach, Werner Israel [3] has shown that the gravitational equations should take the form

$$-(8\pi)^{-1}G^{\alpha\beta} = J^{\alpha\beta} = T^{\alpha\beta} + \frac{1}{2}(S^{\alpha\gamma\beta} + S^{\beta\gamma\alpha} - S^{\alpha\beta\gamma})_{|\gamma} \tag{1.1}$$

for a medium with internal spin acting as source of electromagnetic and gravitational fields. Here $S^{\alpha\gamma\beta}$ is the material spin flux and $T^{\alpha\beta}$ is the energy momentum tensor for non-spinning media (the sum of a material and an electromagnetic tensor). $T^{\alpha\beta}$ is neither symmetric nor conserved, only the total energy momentum tensor $J^{\alpha\beta}$ has these properties. It will be seen that the Lagrangian approach gives exactly the same form for the total tensor $J^{\alpha\beta}$ as found in [3].

However, the first part of the thesis is concerned with the derivation, from an action principle, of equations of motion for a spinning particle moving in given external fields. Such equations are of interest

for two reasons. Firstly for comparison with equations obtained by other methods, and secondly because of their close connection with the field equations for spinning media. The connection can be seen in [3] where we are led to equation (1.1) as a consequence of Maxwell's equations and <u>single</u> particle equations of motion. The single particle equations will in fact be shown to follow from the four dimensional action integral of the field theory by "variation of world lines".

It will also be shown that the Dirac equation in general relativity leads to equations of motion and spin having the same form as the Lagrangian equations.

It is appropriate at this point to introduce some notation and useful identities satisfied by a scalar Lagrangian. (For further elaborations see Appendices one and two.) We consider relative tensor fields $\Phi_A(x)$. Here A denotes both a labelling of the tensor <u>and</u> its collection of contravariant and covariant indices. Thus,

$$\{\Phi_A(x)\} \equiv \{\Phi_{(A_1)\beta_1\ldots\beta_n}^{\alpha_1\ldots\alpha_m} , \Phi_{(A_2)\beta_1\ldots\beta_t}^{\alpha_1\ldots\alpha_s} , \ldots\} .$$

$\Phi^A(x)$ denotes the same tensor with its covariant indices raised and contravariant indices lowered. A repeated capital Latin index, such as in $\Phi^A \Phi_A$, denotes summation over both the tensor indices <u>and</u> the set of tensors to which A refers. Sometimes an index is printed as $\underset{\sim}{A}$ whenever $\underset{\sim}{A}$ and A denote different sets of tensors. For example, $\Phi_{\underset{\sim}{A}}$ may denote both a set of tensor fields Φ_A and the set of covariant derivatives $\Phi_{A|\alpha}$. i.e.

$$\Phi_{\underset{\sim}{A}} \equiv (\Phi_A, \Phi_{A|\alpha}) \quad .$$

Under a co-ordinate transformation $\bar{x}^\mu = \bar{x}^\mu(x^\lambda)$ a relative tensor Φ_A transforms as

$$\bar{\Phi}_A(\bar{x}) = \Lambda_A{}^B(X^\rho{}_\sigma)\, \Phi_B(x) \ , \ X^\rho{}_\sigma \equiv \frac{\partial \bar{x}^\rho}{\partial x^\sigma} \tag{1.2}$$

($\Lambda_A{}^B$ is set equal to zero whenever A and B refer to different tensors, so that the summation becomes a summation over the indices of a <u>single</u> tensor Φ_A). Define infinitesimal generators by

$$(I_A{}^B)_\rho{}^\sigma \equiv \frac{\partial \Lambda_A{}^B}{\partial X^\rho{}_\sigma} \quad (X^\rho{}_\sigma = \delta^\rho_\sigma) \quad . \tag{1.3}$$

The $(I_A{}^B)_\rho{}^\sigma$'s may be constructed explicitly from the recursion formulas (see Appendix one)

$$(I)_\rho{}^\sigma = -w\, \delta^\sigma_\rho \quad \text{for a relative scalar } \phi \text{ of weight } w$$
$$(\text{i.e. } \bar{\phi}(\bar{x}) = \phi(x)\ |\ \partial x/\partial \bar{x}|^w\ ) \ , \tag{1.4}$$

$$(I_{A\alpha}{}^{B\beta})_\rho{}^\sigma = (I_A{}^B)_\rho{}^\sigma\, \delta^\beta_\alpha - \delta^B_A\, \delta^\sigma_\alpha\, \delta^\beta_\rho \ , \tag{1.5}$$

$$(I^{\alpha B}_{A\beta})_\rho{}^\sigma = (I_A{}^B)_\rho{}^\sigma\, \delta^\alpha_\beta + \delta^B_A\, \delta^\alpha_\rho\, \delta^\sigma_\beta \ , \tag{1.6}$$

(where $\delta^B_A = 1$ if A and B both denote the same tensor <u>and</u> the same tensor indices, and is zero otherwise). Useful formulas are

$$\Phi_{A|\tau} = \partial_\tau\, \Phi_A + \Gamma^\rho{}_{\sigma\tau} (I_A{}^B)_\rho{}^\sigma\, \Phi_B \tag{1.7}$$

$$\Phi_{A|\mu\nu} - \Phi_{A|\nu\mu} = -R^\rho{}_{\sigma\mu\nu}\, (I_A{}^B)_\rho{}^\sigma\, \Phi_B \tag{1.8}$$

and

$$\bar{\Phi}_A(\bar{x}) - \Phi_A(x) = (I_A{}^B)_\rho{}^\sigma \, \Phi_B(\partial_\sigma \xi^\rho) \tag{1.9}$$

for the change of Φ_A under infinitesimal co-ordinate transformation $\bar{x}^\rho = x^\rho + \xi^\rho(x)$. It follows from (1.9) that the condition that the function $\underset{\sim}{L}(\underset{\sim}{\Phi_A})$ be a relative scalar of weight w is (see Appendix two)

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{\Phi_A}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma \, \Phi_{\underset{\sim}{B}} + w \, \delta_\rho^\sigma \, \underset{\sim}{L} = 0 \quad . \tag{1.10}$$

Specialising (1.10) to the case of a scalar density $(w=1)$ $\underset{\sim}{L}(\Phi_{\underset{\sim}{A}}) = \underset{\sim}{L}(\Phi_A, \Phi_{A|\alpha})$ depending on a set of relative tensor fields Φ_A and their first covariant derivatives $\Phi_{A|\alpha}$, defining the variational derivative

$$\frac{\delta \underset{\sim}{L}}{\delta \Phi_A} \equiv \underset{\sim}{L}^A - \underset{\sim}{L}^{A\alpha}{}_{|\alpha} \tag{1.11}$$

where

$$\underset{\sim}{L}^A \equiv \frac{\partial \underset{\sim}{L}}{\partial \Phi_A} \quad \text{and} \quad \underset{\sim}{L}^{A\alpha} \equiv \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}} \tag{1.12}$$

and defining tensor densities

$$\underset{\sim}{t}_\rho{}^\sigma \equiv \underset{\sim}{L} \, \delta_\rho^\sigma - \Phi_{A|\rho} \, \underset{\sim}{L}^{A\sigma} \quad ; \quad \underset{\sim}{U}^{\tau\sigma}{}_\rho \equiv \underset{\sim}{L}^{A\tau} (I_A{}^B)_\rho{}^\sigma \, \Phi_B \tag{1.13}$$

gives the identity

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{t}_\rho{}^\sigma + \frac{\delta \underset{\sim}{L}}{\delta \Phi_A} (I_A{}^B)_\rho{}^\sigma \, \Phi_B = 0 \quad . \tag{1.14}$$

CHAPTER TWO

EQUATIONS OF MOTION FOR A SINGLE PARTICLE

§2.1 Introduction.

We will be concerned with the derivation of equations of motion for the momentum P_μ and spin $S_{\mu\nu}$ of a single particle moving in given external fields. In special relativity theory, the natural definition of an angular momentum tensor is

$$M_{\mu\nu} \equiv x_\mu P_\nu - x_\nu P_\mu \quad ,$$

where P_μ is the particle four momentum. $M_{\mu\nu}$ is antisymmetric and in a system of zero three momentum, i.e. $P_1 = P_2 = P_3 = 0$, has only three non-vanishing components,

$$M^{(0)}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & x_1 P^{(0)}_4 \\ 0 & 0 & 0 & x_2 P^{(0)}_4 \\ 0 & 0 & 0 & x_3 P^{(0)}_4 \\ -x_1 P^{(0)}_4 & -x_2 P^{(0)}_4 & -x_3 P^{(0)}_4 & 0 \end{bmatrix}$$

This may be written as

$$M_{[\kappa\lambda} P_{\mu]} = 0 \quad . \tag{2.1.1}$$

We will consider particles which also possess an intrinsic, or internal, angular momentum, independent of a choice of origin, and this "spin" will also be represented by an antisymmetrical tensor, $S_{\mu\nu}$ say. Like the orbital angular momentum, we will assume that the spin is characterised by three components in a system of zero three momentum, however we this time take the space components to be non-vanishing.

$$S^{(0)}_{\mu\nu} = \begin{bmatrix} 0 & S^{(0)}_{12} & S^{(0)}_{13} & 0 \\ -S^{(0)}_{12} & 0 & S^{(0)}_{23} & 0 \\ -S^{(0)}_{13} & -S^{(0)}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $S^{(0)}_{\mu 4} = 0$ when $P_1 = P_2 = P_3 = 0$, $P_4 \neq 0$. i.e. $0 = S^{(0)}_{\mu 4} P^{4(0)} = S^{(0)}_{\mu\nu} P^{\nu(0)} = S_{\mu\nu} P^{\mu}$. So this condition is written as

$$S_{\mu\nu} P^{\nu} = 0 \quad . \tag{2.1.2}$$

Whenever the four velocity u^{λ} is parallel to the four momentum, (2.1.2) may be replaced by the condition

$$S_{\mu\nu} u^{\nu} = 0 \quad . \tag{2.1.3}$$

However, it is usually assumed nowadays that when a particle has intrinsic spin, that the four momentum is no longer parallel to the four velocity. This was first suggested by Weyssenhoff and Raabe.

With P^{λ} no longer parallel to u^{λ} , the conditions $S_{\mu\lambda} u^{\lambda} = 0$ and $S_{\mu\lambda} P^{\lambda} = 0$ have different meanings. For an extended body (see [4])

these are defining equations for the world line of the centre of mass. However, whilst $S_{\mu\lambda} P^{\lambda} = 0$ determines a single world line which under certain assumptions will be a time-like geodesic, $S_{\mu\lambda} u^{\lambda} = 0$ does not determine a unique world line. We therefore take condition (2.1.2) to be valid, this equation determining the velocity u^{λ} in terms of P^{λ} and $S_{\mu\nu}$.

Different methods have been used to derive equations of motion for spinning particles. There is the multipole formalism for an extended body which consists of integration of the conservation identities $0 = T^{\alpha\beta}{}_{|\beta}$ for the total energy tensor over space-like sections of the particle world tube [5]. Equations obtained from a differential geometric viewpoint have been found by Hans Künzle [6], and there are generalisations of rest-frame equations to arbitrary frames [7]. Lagrangian equations have either been obtained by a special choice of Lagrangian or only in the approximation of special relativity [8]. Before the derivation of covariant Lagrangian equations it will be useful to discuss the general form of a Lagrangian describing a spinning particle.

A particle with intrinsic magnetic moment $\underset{\sim}{\mu}$ and intrinsic electric dipole moment $\underset{\sim}{d}$, moving in an external electromagnetic field, has an additional energy $\underset{\sim}{\mu}\cdot\underset{\sim}{H} - \underset{\sim}{d}\cdot\underset{\sim}{E}$ where $\underset{\sim}{H}$ = magnetic field, $\underset{\sim}{E}$ = electric field. We have $\underset{\sim}{\mu}\cdot\underset{\sim}{H} - \underset{\sim}{d}\cdot\underset{\sim}{E} = \frac{1}{2} M^{\mu\nu} F_{\mu\nu}$ where the four tensor $M^{\mu\nu}$ is formed from $\underset{\sim}{\mu}$ and $\underset{\sim}{d}$ in the same way that the electromagnetic field tensor $F_{\mu\nu}$ is formed from $\underset{\sim}{H}$ and $\underset{\sim}{E}$. So, in terms of a Lagrangian L, the magnetic moment tensor for a particle would be defined as $M^{\mu\nu} \equiv 2\, \partial L/\partial F_{\mu\nu}$. We will also assume that the four vector potential

A^μ appears in the Lagrangian only via a coupling to the electric charge current J_μ , so that $J_\mu = \partial L/\partial A^\mu$.

To give the equations for both the trajectory and for the spin, the Lagrangian must also depend on "co-ordinates" or "internal degrees of freedom" corresponding to spin, whose variation leads to the spin equations, just as variation of the space-time co-ordinates leads to the equations of motion.

The spin of a particle may be described by the rotation of a set of axes attached to the particle, so we introduce an orthonormal tetrad defined at each point of the particle world line, $e_\mu^{(a)} = e_\mu^{(a)}(s)$, s = proper time along the world line, (a) goes from 1 to 4, $e_\mu^{(a)}\ e^{(b)\mu} = \eta^{ab}$, $\eta^{ab} = (1,1,1,-1)$. These four four-vectors constitute the "spin co-ordinates" and we consider a Lagrangian that is a function of $u^\lambda = dx^\lambda/ds$, $e_\mu^{(a)}$, $\dot{e}_\mu^{(a)} = \delta\, e_\mu^{(a)}/\delta s$ (absolute derivative), and external fields Φ_A

$$L = L(u^\lambda, e_\mu^{(a)}, \dot{e}_\mu^{(a)}, \Phi_A) \quad .$$

The natural definition of spin in terms of L is

$$S^{\mu\nu} \equiv 2\, e^{(a)[\mu} \frac{\partial L}{\partial \dot{e}^{(a)}_{\nu]}} \quad . \tag{2.1.4}$$

This is the generalisation to relativity of spin in classical mechanics (see Appendix three.)

§2.2 Constraints.

In the derivation of equations we cannot vary all the co-ordinates freely since they are not all independent and must satisfy constraints on variation. To obtain the spin equations we vary the vectors $e^{(a)}_{\mu}$ keeping them orthonormal, i.e., we vary $e^{(a)}_{\mu}$ subject to $e^{(a)}_{\mu}\ e^{(b)\mu} = \eta^{ab}$. The variations $de^{(a)}_{\mu}$ are not all independent and must satisfy $de^{((a)}_{\mu}\ e^{(b))\mu} = 0$, or equivalently $de^{(a)}_{(\mu}\ e_{(a)\nu)} = 0$. The constraints are ten relations between sixteen variables so we have essentially six independent variables.

Introducing scalar Lagrange multipliers $\mu_{ab}(s) = \mu_{ba}(s)$ and $L^{*} = L + \mu_{ab}(e^{(a)}_{\mu}\ e^{(b)\mu} - \eta^{ab})$, then extremising $\int L^{*}\, ds$ for arbitrary variations $de^{(a)}_{\mu}$ vanishing at the endpoints gives

$$\frac{\delta L^{*}}{\delta e^{(a)}_{\mu}} = 0 \quad , \quad \text{i.e.} \quad \frac{\delta L}{\delta e^{(a)}_{\mu}} + 2\,\mu_{ab}\, e^{(b)\mu} = 0 \quad . \qquad (2.2.1)$$

The Lagrange multipliers are therefore

$$\mu_{ac} = -\frac{1}{2}\frac{\delta L}{\delta e^{(a)}_{\mu}}\, e_{(c)\mu} \quad .$$

To eliminate μ_{ab} in (2.2.1) and obtain spin equations, multiplying (2.2.1) by $e^{(a)}_{\nu}$ and antisymmetrising gives

$$\frac{\delta L}{\delta e^{(a)}_{[\mu}}\, e^{(a)}_{\nu]} = -2\,\mu_{ab}(e^{(b)\mu}\ e^{(a)}_{\nu} - e^{(b)}_{\nu}\ e^{(a)\mu})$$

$$= -2\,\mu_{ab}\, e^{(b)\mu}\ e^{(a)}_{\nu} + 2\,\mu_{ab}\, e^{(a)\mu}\ e^{(b)}_{\nu}$$

$$= 0 \quad (\text{since } \mu_{ab} = \mu_{ba}) \quad .$$

These are the six spin dynamical equations.

Variation of the world line $x^\mu(s)$, s = proper time, to obtain equations of motion does not lead simply to $\delta L/\delta x^\mu = 0$ where the action is $I = \int L(x^\mu, dx^\mu/ds)\, ds$ because of the constraint $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$ ($\dot{x}^\mu \equiv dx^\mu/ds$) . We remove this constraint by a process of "parametrisation". Introducing an arbitrary invariant parameter λ , $s = s(\lambda)$, then

$$ds = (-g_{\mu\nu}dx^\mu dx^\nu)^{1/2} = (-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}\, d\lambda = \frac{ds}{d\lambda} \cdot d\lambda$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$. We rewrite I as

$$I = \int L(x^\mu(s), dx^\mu/ds)\, ds$$

$$= \int L(x^\mu(\lambda), \dot{x}^\mu\, d\lambda/ds)\, \frac{ds}{d\lambda} \cdot d\lambda$$

$$= \int \tilde{L}(x^\mu, \dot{x}^\mu, ds/d\lambda)\, d\lambda$$

defining a new Lagrangian $\tilde{L}(x^\mu, \dot{x}^\mu, ds/d\lambda)$. The variables of $\tilde{L}$ are still constrained since we now have $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -(\frac{ds}{d\lambda})^2$ replacing $g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -1$. We eliminate the constraint by using the above equation to replace $ds/d\lambda$ in $\tilde{L}$. Then

$$I = \int \tilde{L}(x^\mu, \dot{x}^\mu, (-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2})\, d\lambda$$

$$= \int L_1(x^\mu, \dot{x}^\mu)\, d\lambda$$

where L_1 is defined as

$$L_1(x^\mu,\dot{x}^\mu) = \tilde{L}(x^\mu,\dot{x}^\mu,(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2})$$

$$= (-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}\ L(x^\mu,(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{-1/2}\ \dot{x}^\mu) \quad .$$

The Lagrangian L_1 is homogeneous of degree one in $\dot{x}^\mu$, irrespective of the dependence of $L(x^\mu,dx^\mu/ds)$ upon dx^μ/ds.

Our action principle will henceforth be

$$0 = \delta I = \delta \int L_1(x^\mu,\dot{x}^\mu)d\lambda$$

for arbitrary variations in x^μ (no constraint) where λ is any invariant parameter and L_1 is homogeneous of degree one in $\dot{x}^\mu$.

§2.3 Covariant Equations of Motion in Given External Fields.

The equations are obtained from the action principle $\delta I = \delta \int L_1\, d\lambda = 0$ for arbitrary variations of $x^\mu(\lambda)$ and $e^{(a)}_\mu(\lambda)$ with fixed endpoints $x^\mu(\lambda_i)$, $e^{(a)}_\mu(\lambda_i)$, $(i = 1,2)$. λ is any invariant parameter, $\lambda = \lambda(s)$, and L_1 is homogeneous of degree one in $v^\mu \equiv dx^\mu/d\lambda$.

$$L_1 = L_1(v^\mu,e^{(a)}_\mu,\delta e^{(a)}_\mu/\delta s,\Phi_{\underset{\sim}{A}}) \tag{2.3.1}$$

The set $\underset{\sim}{\Phi_A}$ consists of the external fields ϕ_A, the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ and their covariant derivatives,

$$\Phi_{\underset{\sim}{A}} = (\phi_A,R^\alpha{}_{\beta\gamma\delta},\phi_{A|\alpha},R^\alpha{}_{\beta\gamma\delta|\mu},\ldots) \quad .$$

Defining $\dot{e}_\mu^{(a)} \equiv \delta e_\mu^{(a)}/\delta\lambda$, then

$$\delta e_\mu^{(a)}/\delta s = \dot{e}_\mu^{(a)}\, d\lambda/ds = \dot{e}_\mu^{(a)}\, (-\eta_{bc} e_\alpha^{(b)} e_\beta^{(c)} v^\alpha v^\beta)^{-1/2}$$

so L_1 may be reexpressed as a new function

$$L_1 = L(v^\mu, e_\mu^{(a)}, \dot{e}_\mu^{(a)}, \Phi_{\underset{\sim}{A}}) \quad . \tag{2.3.2}$$

We define the canonical momentum P_μ , the spin angular momentum $S^{\rho\sigma} = S^{[\rho\sigma]}$ and the multipole moments by

$$P_\mu = \frac{\partial L}{\partial v^\mu} \quad , \quad S_\rho{}^\sigma = 2\, e^{(a)}_{[\rho} \frac{\partial L}{\partial \dot{e}^{(a)}_{\sigma]}} \quad , \quad M^{\underset{\sim}{A}} = \frac{\partial L}{\partial \Phi_{\underset{\sim}{A}}} \tag{2.3.3}$$

(the derivatives $\partial L/\partial \Phi_{\underset{\sim}{A}} = M^{\underset{\sim}{A}}$ are defined such that $dL = M^{\underset{\sim}{A}}\, d\Phi_{\underset{\sim}{A}}$ and $M^{\underset{\sim}{A}}$ has the same <u>algebraic symmetries</u> as $\Phi_{\underset{\sim}{A}}$. For example, if $L = F_{\mu\nu}\, X^{\mu\nu}$ with $F_{\mu\nu}$ antisymmetric, then $L = F_{\mu\nu}\, X^{[\mu\nu]}$ and $\partial L/\partial F_{\alpha\beta}$ is defined as $X^{[\alpha\beta]}$).

In terms of L_1 we have

$$M^{\underset{\sim}{A}} = \frac{\partial L_1}{\partial \Phi_{\underset{\sim}{A}}} \quad , \quad S_\rho{}^\sigma = 2\, e^{(a)}_{[\rho} \frac{\partial L_1}{\partial \frac{\delta e^{(a)}_{\sigma]}}{\delta s}} \cdot \frac{d\lambda}{ds} \tag{2.3.4}$$

$$\begin{aligned} P_\mu &= \frac{\partial L_1}{\partial v^\mu} + \frac{\partial L_1}{\partial \frac{\delta e^{(a)}_\alpha}{\delta s}} \frac{\partial}{\partial v^\mu} \left(\frac{\delta e^{(a)}_\alpha}{\delta s}\right) \\ &= \frac{\partial L_1}{\partial v^\mu} + \frac{\partial L_1}{\partial \frac{\delta e^{(a)}_\alpha}{\delta s}}\, \dot{e}^{(a)}_\alpha\, v_\mu \left(\frac{d\lambda}{ds}\right)^3 \quad . \end{aligned} \tag{2.3.5}$$

Since L_1 is homogeneous degree one in v^μ we have $S_\rho{}^\sigma$ and P_μ both

independent of parametrisation, and $\underset{\sim}{M}^{A}$ proportional to $ds/d\lambda$.

A variation $de_{\mu}^{(a)}$ in the tetrad, keeping the world line fixed, gives (since $g_{\mu\nu}$ and $\Gamma_{\mu\ \nu}^{\ \lambda}$ are unaltered) the variation of $\dot{e}_{\mu}^{(a)}$ to be

$$d\left(\frac{\delta e_{\mu}^{(a)}}{\delta\lambda}\right) = d\left(\frac{de_{\mu}^{(a)}}{d\lambda} - \Gamma_{\mu\ \lambda}^{\ \beta}\, e_{\beta}^{(a)}\, v^{\lambda}\right) = \frac{\delta}{\delta\lambda}\left(de_{\mu}^{(a)}\right) .$$

Hence

$$dL = \frac{\partial L}{\partial e_{\mu}^{(a)}}\, de_{\mu}^{(a)} + \frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\, \frac{\delta}{\delta\lambda}\left(de_{\mu}^{(a)}\right)$$

$$= \left(\frac{\partial L}{\partial e_{\mu}^{(a)}} - \frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\right)\right) de_{\mu}^{(a)} + \frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\, de_{\mu}^{(a)}\right) .$$

$\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\, de_{\mu}^{(a)}$ is a scalar, so $\frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\, de_{\mu}^{(a)}\right) = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\, de_{\mu}^{(a)}\right)$. The spin equations are therefore

$$\frac{\delta L}{\delta e_{[\mu}^{(a)}}\, e^{(a)}_{\ \nu]} = 0 \quad \text{where} \quad \frac{\delta L}{\delta e_{\mu}^{(a)}} = \frac{\partial L}{\partial e_{\mu}^{(a)}} - \frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial \dot{e}_{\mu}^{(a)}}\right) . \quad (2.3.6)$$

Identity (1.10) with $w = 0$ for the particle Lagrangian is

$$P_{\mu}v^{\nu} - \frac{\partial L}{\partial e_{\nu}^{(a)}}\, e_{\mu}^{(a)} - \frac{\partial L}{\partial \dot{e}_{\nu}^{(a)}}\, \dot{e}_{\mu}^{(a)} + \underset{\sim}{M}^{A}(\underset{\sim}{I}_{\underset{\sim}{A}}^{\ \underset{\sim}{B}})_{\mu}^{\ \nu}\, \Phi_{\underset{\sim}{B}} = 0 . \quad (2.3.7)$$

Rearranging and antisymmetrising, this is

$$P^{[\mu}{}_{v}{}^{\nu]} - e^{(a)[\mu}\, \frac{\delta L}{\delta e^{(a)}_{\ \nu]}} - \frac{\delta}{\delta\lambda}\left(e^{(a)[\mu}\, \frac{\partial L}{\partial \dot{e}^{(a)}_{\ \nu]}}\right) + \underset{\sim}{M}^{A}(\underset{\sim}{I}_{\underset{\sim}{A}}^{\ \underset{\sim}{B}})^{[\mu\nu]}\Phi_{\underset{\sim}{B}} = 0 .$$

Spin equations (2.3.6), together with definition (2.3.3) for $S_{\rho}^{\ \sigma}$ and the above identity then give

$$\boxed{\frac{1}{2}\frac{\delta S^{\mu\nu}}{\delta\lambda} = P^{[\mu}{}_{v}{}^{\nu]} + M^{\underset{\sim}{A}}\,(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\mu\nu]}\,\Phi_{\underset{\sim}{B}}} \quad . \tag{2.3.8}$$

To obtain momentum equations we make an arbitrary variation in the world line $dx^\mu(\lambda)$ ($dx^\mu = 0$ at endpoints) keeping $e_\mu^{(a)}$ covariantly constant on variation, $\delta e_\mu^{(a)}(\lambda) = 0$. The most elegant approach is a manifestly covariant one.

The change in L is $dL = \delta L$

$$= \frac{\partial L}{\partial v^\mu}\,\delta(v^\mu) + \frac{\partial L}{\partial e_\mu^{(a)}}\,\delta e_\mu^{(a)} + \frac{\partial L}{\partial \dot{e}_\mu^{(a)}}\,\delta(\dot{e}_\mu^{(a)}) + \frac{\partial L}{\partial \Phi_{\underset{\sim}{A}}}\,\delta\Phi_{\underset{\sim}{A}}$$

where $\delta(v^\mu)$ etc. are the <u>absolute</u> changes in each dynamical variable.

$$\delta(v^\mu) = \delta\left(\frac{dx^\mu}{d\lambda}\right) = d\left(\frac{dx^\mu}{d\lambda}\right) + \Gamma_\nu{}^\mu{}_\gamma\,\frac{dx^\nu}{d\lambda}\,dx^\gamma = \frac{\delta}{\delta\lambda}\,(dx^\mu) \quad .$$

$$\delta e_\mu^{(a)} = 0 \quad , \quad \delta\,\Phi_{\underset{\sim}{A}} = \Phi_{\underset{\sim}{A}|\mu}\,dx^\mu \quad .$$

$$\delta(\dot{e}_\mu^{(a)}) = \delta\left(\frac{\delta e_\mu^{(a)}}{\delta\lambda}\right) = -R_{\mu\gamma\rho\sigma}\,e^{(a)\gamma}\,\frac{dx^\rho}{d\lambda}\,dx^\sigma$$

$$\frac{\partial L}{\partial v^\mu}\,\delta(v^\mu) = \frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial v^\mu}\,dx^\mu\right) - \frac{\delta}{\delta\lambda}\left(\frac{\partial L}{\partial v^\mu}\right)dx^\mu$$

$$= \frac{d}{d\lambda}\,(P_\mu\,dx^\mu) - \frac{\delta P_\mu}{\delta\lambda}\,dx^\mu \quad \text{since } P_\mu\,dx^\mu \text{ is a scalar}$$

$$\therefore \quad \int dL\,d\lambda = \int \left(-\frac{\delta P_\mu}{\delta\lambda} - R_{\alpha\beta\gamma\mu}\,e^{(a)\beta}\,\frac{\partial L}{\partial \dot{e}_\alpha^{(a)}}\,v^\gamma + M^{\underset{\sim}{A}}\,\Phi_{\underset{\sim}{A}|\mu}\right)dx^\mu\,d\lambda \quad .$$

By definition (2.3.3) for $S_\rho{}^\sigma$, the momentum equations are therefore

$$\boxed{\frac{\delta P_\mu}{\delta\lambda} = \frac{1}{2} R_{\alpha\beta\gamma\mu} S^{\alpha\beta} v^\gamma + M^{\underset{\sim}{A}} \Phi_{\underset{\sim}{A}|\mu}} \quad . \tag{2.3.9}$$

In the first term we see the interaction between the spin and the gravitational field.

Together with (2.3.8) and (2.3.9) we need an equation determining v^λ which we take to be $S^{\mu\nu} P_\nu = 0$. In the simplest case with $\Phi_{\underset{\sim}{A}} = \{0\}$, using (2.3.8) and (2.3.9) gives

$$0 = \frac{\delta}{\delta\lambda} (S^{\alpha\beta} P_\beta) = 2P^{[\alpha}v^{\beta]}P_\beta + S^{\alpha\beta} (-\frac{1}{2} R_{\beta\gamma\rho\mu} v^\gamma S^{\rho\mu})$$

$$\text{i.e.} \quad P^\alpha(v^\beta P_\beta) - v^\alpha(P^\beta P_\beta) - \frac{1}{2} S^{\alpha\beta} R_{\beta\gamma\rho\mu} v^\gamma S^{\rho\mu} = 0 \quad . \tag{2.3.10}$$

$S_{\mu\nu} S^{\mu\nu}$ is conserved since

$$\frac{\delta}{\delta\lambda} (S_{\mu\nu}S^{\mu\nu}) = 2 S_{\mu\nu} \frac{\delta S^{\mu\nu}}{\delta\lambda} = 2 S_{\mu\nu}(P^\mu v^\nu - P^\nu v^\mu) = 0 \quad .$$

§2.4 Motion in an External Electromagnetic Field.

As a special case of §2.3, take $\Phi_{\underset{\sim}{A}} = \{A_\alpha, A_{\alpha|\beta}, R_B\}$ where R_B denotes $R^\alpha{}_{\beta\gamma\delta}$ and A_α is the four vector potential. Assume that $A_{\alpha|\beta}$ appears in the Lagrangian only through $F_{\beta\alpha} = A_{\alpha|\beta} - A_{\beta|\alpha}$. We define the magnetic moment as

$$M^{\alpha\beta} = \frac{\partial L}{\partial A_{\beta|\alpha}} = 2 \frac{\partial L}{\partial F_{\alpha\beta}} \quad ,$$

and the quadrupole gravitational moment tensor as $Q^A = Q_\alpha{}^{\beta\gamma\delta} = \partial L/\partial R^\alpha{}_{\beta\gamma\delta}$ and we assume the dependence of L upon A_α is given by $\partial L/\partial A_\alpha \equiv J^\alpha = e\, v^\alpha$. (e = electric charge).

$$M^{\underset{\sim}{A}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma\, \Phi_{\underset{\sim}{B}}$$

$$= \frac{\partial L}{\partial A_\alpha}(-\delta^\sigma_\alpha \delta^\beta_\rho)A_\beta + \frac{\partial L}{\partial A_{\alpha_1|\alpha_2}}(-\delta^{\beta_1}_{\alpha_1}\delta^\sigma_{\alpha_2}\delta^{\beta_2}_\rho - \delta^\sigma_{\alpha_1}\delta^{\beta_1}_\rho\delta^{\beta_2}_{\alpha_2})A_{\beta_1|\beta_2} + Q^A(I_A{}^B)_\rho{}^\sigma R_B$$

$$= -\frac{\partial L}{\partial A_\sigma}A_\rho - \frac{\partial L}{\partial A_{\alpha_1|\sigma}}A_{\alpha_1|\rho} - \frac{\partial L}{\partial A_{\sigma|\alpha_2}}A_{\rho|\alpha_2} + Q^A(I_A{}^B)_\rho{}^\sigma R_B$$

$$= -e\, v^\sigma A_\rho - M^{\sigma\alpha}F_{\rho\alpha} + Q_\rho{}^{\beta\gamma\delta}R^\sigma{}_{\beta\gamma\delta} - 3\, Q_\alpha{}^{\sigma\gamma\delta}R^\alpha{}_{\rho\gamma\delta}$$

$$\therefore \quad M^{\underset{\sim}{A}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\rho\sigma]}\Phi_{\underset{\sim}{B}} = -e\, A^{[\rho}v^{\sigma]} - F^{[\rho}{}_\alpha M^{\sigma]\alpha} - 4\, R^{[\rho}{}_{\alpha\beta\gamma}Q^{\sigma]\alpha\beta\gamma} \; .$$

Substitution of the above into (2.3.8) and defining a new "kinetic" momentum $p_\mu \equiv P_\mu - e\, A_\mu$, gives spin equations

$$\boxed{\frac{1}{2}\frac{\delta S^{\mu\nu}}{\delta\lambda} = p^{[\mu}v^{\nu]} - F^{[\mu}{}_\alpha M^{\nu]\alpha} - 4\, R^{[\mu}{}_{\alpha\beta\gamma}Q^{\nu]\alpha\beta\gamma}} \qquad (2.4.1)$$

$$\frac{\delta P_\mu}{\delta\lambda} - M^{\underset{\sim}{A}}\Phi_{\underset{\sim}{A}|\mu} = \frac{\delta p_\mu}{\delta\lambda} + e\, A_{\mu|\nu}v^\nu - \frac{\partial L}{\partial A_\nu}A_{\nu|\mu} - \frac{\partial L}{\partial A_{\lambda|\nu}}A_{\lambda|\nu|\mu} - Q^A R_{A|\mu}$$

$$= \frac{\delta p_\mu}{\delta\lambda} + e\, F_{\nu\mu}v^\nu - \frac{1}{2}M^{\nu\lambda}F_{\nu\lambda|\mu} - Q_\alpha{}^{\beta\gamma\delta}R^\alpha{}_{\beta\gamma\delta|\mu} \; .$$

The momentum equations (2.3.9) are therefore

$$\boxed{\frac{\delta p_\mu}{\delta\lambda} = \frac{1}{2}R_{\alpha\beta\gamma\mu}S^{\alpha\beta}v^\gamma + e\, F_{\mu\nu}v^\nu + \frac{1}{2}M^{\nu\lambda}F_{\nu\lambda|\mu} + Q_\alpha{}^{\beta\gamma\delta}R^\alpha{}_{\beta\gamma\delta|\mu}} \qquad (2.4.2)$$

In the absence of <u>any</u> fields, $R_{\alpha\beta\gamma\delta} = 0 = F_{\alpha\beta}$, $\frac{\delta}{\delta\lambda} = \frac{d}{d\lambda}$, (2.4.1) and (2.4.2) reduce to $dS^{\mu\nu}/d\lambda = p^{\mu}v^{\nu} - p^{\nu}v^{\mu}$ and $dp_{\mu}/d\lambda = 0$, and (2.3.10) gives p_{μ} parallel to v_{μ} . This implies $dS^{\mu\nu}/d\lambda = 0$ and $dJ^{\mu\nu}/d\lambda = 0$ where $J^{\mu\nu}$ (orbital angular momentum) is $J^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$.

Now choosing λ to be proper time, $\lambda = s$, $v^{\mu}v_{\mu} = -1$, we can multiply (2.4.1) by v_{ν} and solve for p^{μ} (set $Q_{\alpha}{}^{\beta\gamma\delta} = 0$ for simplicity). We obtain

$$p^{\mu} = m_o v^{\mu} - 2\, F^{[\mu}{}_{\alpha}M^{\nu]\alpha}\, v_{\nu} - \frac{\delta S^{\mu\nu}}{\delta s}\, v_{\nu} \tag{2.4.3}$$

$(m_o \equiv -p^{\nu}v_{\nu})$. Notice that p^{μ} is not parallel to v^{μ} , p^{μ} is expressed in the above equation as a sum of a term parallel to v^{μ} (i.e. $m_o v^{\mu}$) and a term orthogonal to v^{μ} . A satisfactory feature is that the energy momentum vector p^{μ} should also depend on the spin.

Now replacing p^{μ} by the right hand side of (2.4.3) in (2.4.1) gives

$$\frac{\delta S^{\mu\nu}}{\delta s} = -2\, v_{\lambda}\, \frac{\delta S^{[\mu\lambda}}{\delta s}\, v^{\nu]} - 2v_{\lambda}F^{[\mu}{}_{\alpha}M^{\lambda\alpha}v^{\nu]} + 2v_{\lambda}F^{\lambda}{}_{\alpha}M^{[\mu\alpha}v^{\nu]} - 2F^{[\mu}{}_{\alpha}M^{\nu]\alpha}$$

i.e.

$$\frac{\delta S^{\mu\nu}}{\delta s} - 2v_{\lambda}\, \frac{\delta S^{\lambda[\mu}}{\delta s}\, v^{\nu]} = 2v_{\lambda}M^{\lambda\alpha}F_{\alpha}{}^{[\mu}v^{\nu]} + 2v_{\lambda}F^{\lambda\alpha}M^{[\mu}{}_{\alpha}v^{\nu]} - 2F^{[\mu}{}_{\alpha}M^{\nu]\alpha} \quad .$$

In the case $M^{\alpha\beta} = K(s)\, S^{\alpha\beta}$, $Q_{\alpha}{}^{\beta\gamma\delta} = 0$, we have the spin conservation equation $\frac{d}{ds}(S_{\alpha\beta}S^{\alpha\beta}) = 0$ from (2.4.1).

In summary, sections (2.3) and (2.4) contain covariant Lagrangian equations of motion for spinning particles in gravitational and any other external fields. The ease of derivation and their generality are features of equations (2.3.8) and (2.3.9). They are general in the sense that no explicit dependence upon its variables is assumed for the Lagrangian, L is <u>any</u> function of v^λ, $e_\mu^{(a)}$ etc. The only possibility for obtaining different equations might be by use of different definitions of spin, four momentum etc. (though the <u>content</u> of the equations would not change). For example one may wish to define a new momentum in (2.4.1) and (2.4.2) by $P_\mu = p_\mu + F_{\mu\lambda} q^\lambda$ where q^λ is the electric dipole moment. (2.4.1) and (2.4.2) then become (with $Q_\alpha{}^{\beta\gamma\delta} = 0$)

$$\frac{\delta P_\mu}{\delta\lambda} = -\frac{1}{2} R_{\mu\gamma\alpha\beta} S^{\alpha\beta} v^\gamma + e F_{\mu\nu} v^\nu + \frac{1}{2} M^{\nu\lambda} F_{\nu\lambda|\mu} + \frac{\delta}{\delta\lambda} (F_{\mu\lambda} q^\lambda)$$

$$\frac{\delta S^{\mu\nu}}{\delta\lambda} = 2P^{[\mu} v^{\nu]} - 2F^{[\mu}{}_\alpha (M^{\nu]\alpha} + v^{\nu]} q^\alpha) \quad .$$

These are the equations obtained by Suttorp and de Groot [9] in the special relativistic case.

CHAPTER THREE

RELATIVISTIC FIELD EQUATIONS FOR POLARIZED MEDIA

§3.1 We consider a polarised dustlike medium interacting with the gravitational field and any other fields present. The medium is taken to consist of coherently moving, spinning particles which interact only through the external fields. The particle world lines are the integral curves of a numerical flux vector field $n\, u^{\alpha}(x)$, $u_{\alpha}u^{\alpha} = -1$, describing the medium, which is assumed to satisfy $(nu^{\alpha})_{|\alpha} = 0$ (conservation of number). The internal spin of the medium is described by an orthonormal tetrad field $e_{\alpha}^{(a)}(x)$ which also serves to specify the metric field.

Variation of a four dimensional action integral leads to gravitational equations and equations for the spin, by variation of the tetrad field $e_{\alpha}^{(a)}(x)$. The resultant spin (field) equations are equivalent to the single particle spin equations of motion, and by a variation of world lines, the four dimensional action principle also gives the single particle translational equations of motion. Non-gravitational fields present will of course have field equations obtained by variation w.r. to their components. The outcome is thus a complete set of field equations relating the fields to their sources and also the single particle equations obtained in chapter two.

The equations are obtained from the action principle $\delta I = 0$ for arbitrary variations in the dynamical variables, I being a four dimensional action integral,

$$I = \int (-(16\pi)^{-1}(-g)^{1/2}R + \underset{\sim}{L}(\underset{\sim}{\Phi}_A, \underset{\sim}{\Phi}_{A|\alpha}))d^4x \quad . \tag{3.1.1}$$

We have taken $-(16\pi)^{-1}(-g)^{1/2}R$, where R is the curvature scalar, for the gravitational free field Lagrangian. The scalar density $\underset{\sim}{L}$ is a function of the fields

$$\underset{\sim}{\Phi}_A = \{\underset{\sim}{N}^\alpha, e_\alpha^{(a)}, \psi_A, R^\alpha{}_{\beta\gamma\delta}\} \tag{3.1.2}$$

where $\underset{\sim}{N}^\alpha = \sqrt{-g}\, nu^\alpha$ is the numerical flux, $e_\alpha^{(a)}$ is the tetrad field, $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor and ψ_A an arbitrary set of fields. We are assuming, for simplicity, that $\underset{\sim}{L}$ does not depend on second and higher covariant derivatives of the $\underset{\sim}{\Phi}_A$ (the general case with dependence on higher derivatives is discussed in Chapter Four). We also assume that no derivatives of $R^\alpha{}_{\beta\gamma\delta}$ and $\underset{\sim}{N}^\alpha$ appear in $\underset{\sim}{L}$.

Before proceeding, a few comments on the role of the tetrad field is in order. Firstly, the components $e_\alpha^{(a)}(x)$ are the gravitational variables just as any orthonormal tetrad field describes the metric $g_{\mu\nu} = \eta_{ab}\, e_\mu^{(a)} e_\nu^{(b)}$. Secondly, as a set of spin axes for each world line they are spin variables that will be dynamically determined along each world line in accordance with the particle spin equations of motion. Because of this second role the Lagrangian has an explicit dependence upon the $e_\alpha^{(a)}$ in addition to its dependence via the metric $g_{\mu\nu}$ (so that we could not replace $e_\alpha^{(a)}$ by $g_{\mu\nu}$ in (3.1.1) and (3.1.2)) and hence $\underset{\sim}{L}$ would not be expected to be invariant under an arbitrary field of tetrad rotations. One could have considered a Lagrangian density depending upon both $g_{\mu\nu}$ and $e_\alpha^{(a)}$ (and then varied $g_{\mu\nu}$ to obtain the ten gravitational equations and varied six independent components of the tetrad to obtain

spin equations). However, since the orthonormality conditions $g_{\mu\nu} = \eta_{ab} e^{(a)}_{\mu} e^{(b)}_{\nu}$ are ten relations between the twenty six variables $g_{\mu\nu}$, $e^{(a)}_{\alpha}$, we have sixteen independent variables and it is much simpler to take the $e^{(a)}_{\alpha}$ as our sixteen. We take, therefore, arbitrary <u>independent</u> variations in the $e^{(a)}_{\alpha}$ to obtain ten gravitational equations and six spin equations.

Much guidance for the following calculation is due to Rosenfeld (see [10]) (and also Belinfante [11]). The difference between their work and the present is that spin is considered in their scheme only in the sense that the fields ψ_A are allowed to be spinor fields.

§3.2 <u>Tetrad Variation</u>.

We make an arbitrary variation $de^{(a)}_{\mu}$ in the tetrad field. The resultant variations in functions of $e^{(a)}_{\mu}$ are (see Appendix four for details)

$$dg_{\alpha\beta} = 2\,\eta_{ab} e^{(a)}_{(\alpha}\; de^{(b)}_{\beta)} \quad \text{for the metric} \quad , \tag{3.2.1}$$

$$d\,\Gamma^{\rho}{}_{\sigma\tau} = dg^{\rho}{}_{(\sigma|\tau)} - \frac{1}{2}\, dg_{\sigma\tau}{}^{|\rho} \quad \text{for the Christoffel symbols,} \tag{3.2.2}$$

$$dR^{\alpha}{}_{\beta\gamma\delta} = 2\, d\Gamma^{\alpha}{}_{\beta[\delta|\gamma]} \quad \text{for the Riemann tensor} \quad . \tag{3.2.3}$$

It follows from (3.2.3) that for an arbitrary tensor density $A_{\alpha}{}^{\beta\gamma\delta}$ that

$$A_\alpha{}^{\beta\gamma\delta}\, dR^\alpha{}_{\beta\gamma\delta} = -2\, A_\alpha{}^{\beta[\gamma\delta]}{}_{|\gamma}\, d\Gamma^\alpha{}_{\beta\delta} + (\text{div}) \tag{3.2.4}$$

where (div) represents a divergence term (which by Gauss's theorem, will play no part in the derivation). From (3.2.2) it follows that for an arbitrary tensor density $B^{\tau\sigma}{}_\rho$ that

$$B^{\tau\sigma}{}_\rho\, d\Gamma^\rho{}_{\sigma\tau} = \frac{1}{2}\,(B^{\rho[\sigma\tau]} + B^{\sigma[\rho\tau]} - B^{\tau(\rho\sigma)})_{|\tau}\, dg_{\rho\sigma} + (\text{div})\ . \tag{3.2.5}$$

The variation in the free field gravitational Lagrangian is

$$(\text{div}) + (-16\pi)^{-1}\, d((-g)^{1/2}R) = (16\pi)^{-1}(-g)^{1/2}\, G^{\alpha\beta}\, dg_{\alpha\beta}$$

$$= (8\pi)^{-1}(-g)^{1/2}\, G^{\alpha\beta}\, e_{(a)\alpha}de_\beta{}^{(a)}\ . \tag{3.2.6}$$

Care must be taken in calculating the change $d\underset{\sim}{L}$ in $\underset{\sim}{L}$ since $\underset{\sim}{L}$ depends upon the $e_\mu{}^{(a)}$ in three ways:

(i) $\underset{\sim}{L}$ has an explicit dependence on $e_\mu{}^{(a)}$ and $e_{\mu|\nu}{}^{(a)}$;

(ii) $\underset{\sim}{L}$ depends on $R^\alpha{}_{\beta\gamma\delta}$;

(iii) $\underset{\sim}{L}$ depends on the $\Gamma^\alpha{}_{\beta\gamma}$'s hidden in covariant derivatives $\Phi_{\underset{\sim}{A}|\alpha}$ (i.e. $e_{\alpha|\beta}{}^{(a)}$ and $\psi_{A|\alpha}$).

The change in $\underset{\sim}{L}(\Phi_{\underset{\sim}{A}},\Phi_{\underset{\sim}{A}|\alpha})$ due to (iii) is $(\partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}|\alpha})\,\dfrac{\partial(\Phi_{\underset{\sim}{A}|\alpha})}{\partial\Gamma^\rho{}_{\sigma\tau}}\, d\Gamma^\rho{}_{\sigma\tau}$ which by (1.7) is $(\partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}|\tau})(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma\,\Phi_{\underset{\sim}{B}}\, d\Gamma^\rho{}_{\sigma\tau}$. Defining

$$\underset{\sim}{L}^{\underset{\sim}{A}} = \partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}}\ ,\quad \underset{\sim}{L}^{\underset{\sim}{A}\alpha} = \partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}|\alpha}\ ,$$

$$\frac{\delta\underset{\sim}{L}}{\delta\Phi_{\underset{\sim}{A}}} = \underset{\sim}{L}^{\underset{\sim}{A}} - \underset{\sim}{L}^{\underset{\sim}{A}\alpha}{}_{|\alpha}\ ,\quad \underset{\sim}{U}^{\tau\sigma}{}_\rho = \underset{\sim}{L}^{\underset{\sim}{A}\tau}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma\,\Phi_{\underset{\sim}{B}}$$

as in Chapter One, and defining $\underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta} = \partial\underset{\approx}{L}/\partial R^{\alpha}{}_{\beta\gamma\delta}$, then the change in $\underset{\sim}{L}$ is

$$d\underset{\sim}{L} = \frac{\delta\underset{\sim}{L}}{\delta e_{\mu}{}^{(a)}}\, de_{\mu}{}^{(a)} + \underset{\sim}{U}^{\tau\sigma}{}_{\rho}\, \delta\Gamma^{\rho}{}_{\sigma\tau} + \underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta}\, dR^{\alpha}{}_{\beta\gamma\delta} + \left(\frac{\partial\underset{\sim}{L}}{\partial e_{\sigma|\alpha}{}^{(a)}}\, de_{\sigma}{}^{(a)}\right)_{|\alpha} \qquad (3.2.7)$$

$$= \frac{\delta\underset{\sim}{L}}{\delta e_{\mu}{}^{(a)}}\, de_{\mu}{}^{(a)} + \tfrac{1}{2}(\underset{\sim}{U}^{\rho[\sigma\tau]} + \underset{\sim}{U}^{\sigma[\rho\tau]} - \underset{\sim}{U}^{\tau(\rho\sigma)})_{|\tau}\, dg_{\rho\sigma}$$

$$- 2\, \underset{\sim}{Q}_{\alpha}{}^{\beta[\gamma\delta]}{}_{|\gamma}\, d\Gamma^{\alpha}{}_{\beta\delta} + (\mathrm{div}) \qquad (3.2.8)$$

using (3.2.4) and (3.2.5) . Antisymmetry of $\underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta}$ in the first two and last two indices and another use of (3.2.5) gives

$$\underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta}{}_{|\gamma}\, d\Gamma^{\alpha}{}_{\beta\delta} = -\underset{\sim}{Q}^{\tau(\rho\sigma)\gamma}{}_{|\gamma\tau}\, dg_{\rho\sigma} + (\mathrm{div}) \quad .$$

Using (3.2.1), (3.2.6) and (3.2.8) the action principle $\delta I = 0$ for any $de_{\mu}{}^{(a)}$ therefore gives

$$-(8\pi)^{-1}\, (-g)^{1/2}\, G^{\alpha\nu}\, e_{(a)\alpha} = \frac{\delta\underset{\approx}{L}}{\delta e_{\nu}{}^{(a)}} + (\underset{\sim}{U}^{\rho[\nu\tau]} + \underset{\sim}{U}^{\nu[\rho\tau]} - \underset{\sim}{U}^{\tau(\rho\nu)})_{|\tau}\, e_{(a)\rho}$$

$$+ 4\underset{\sim}{Q}^{\tau(\rho\nu)\gamma}{}_{|\gamma\tau}\, e_{(a)\rho} \qquad (3.2.9)$$

or equivalently,

$$-(8\pi)^{-1}(-g)^{1/2}G^{\mu\nu} = \frac{\delta\underset{\approx}{L}}{\delta e_{\nu}{}^{(a)}}\, e^{(a)\mu} + (\underset{\sim}{U}^{\mu[\nu\tau]} + \underset{\sim}{U}^{\nu[\mu\tau]} - \underset{\sim}{U}^{\tau(\mu\nu)})_{|\tau}$$

$$+ 4\, \underset{\sim}{Q}^{\tau(\mu\nu)\gamma}{}_{|\gamma\tau} \quad . \qquad (3.2.10)$$

The above are the gravitational and the spin equations. Since every term

is symmetric in μ,ν , except for $e^{(a)\mu} \frac{\delta \underset{\sim}{L}}{\delta e_{\nu}^{(a)}}$, the six spin equations of motion are

$$e^{(a)[\mu} \frac{\delta \underset{\sim}{L}}{\delta e^{(a)}_{\ \ \nu]}} = 0 \quad . \tag{3.2.11}$$

The next step is to replace $e^{(a)\mu} \frac{\delta \underset{\sim}{L}}{\delta e_{\nu}^{(a)}}$ in (3.2.10) by terms having possible physical interpretation and this is achieved by means of the identity of the type (1.14) that $\underset{\sim}{L}$, as a scalar density, must satisfy. Recalling (1.14) which reads

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{t}_{\rho}{}^{\sigma} + \frac{\delta \underset{\sim}{L}}{\delta \Phi_{\underset{\sim}{A}}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \Phi_{\underset{\sim}{B}} = 0 \tag{3.2.12}$$

$$(\underset{\sim}{t}_{\rho}{}^{\sigma} \equiv \underset{\sim}{L}\, \delta^{\sigma}_{\rho} - \Phi_{\underset{\sim}{A}|\rho}\, \underset{\sim}{L}^{A\sigma}) \tag{3.2.13}$$

in the present context with $\underset{\sim}{L} = \underset{\sim}{L}(\Phi_{\underset{\sim}{A}},\Phi_{\underset{\sim}{A}|\alpha})$,

$$\Phi_{\underset{\sim}{A}} = \{\underset{\sim}{N}^{\alpha}, e^{(a)}_{\alpha}, \psi_{A}, R^{\alpha}{}_{\beta\gamma\delta}\} \ , \ \Phi_{\underset{\sim}{A}|\alpha} = \{e^{(a)}_{\alpha|\beta}, \psi_{A|\alpha}\} \quad , \tag{3.2.14}$$

we have

$$\frac{\delta \underset{\sim}{L}}{\delta \Phi_{\underset{\sim}{A}}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \Phi_{\underset{\sim}{B}} = \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\sigma} - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha} \delta^{\sigma}_{\rho} - \frac{\delta \underset{\sim}{L}}{\delta e_{\sigma}^{(a)}} e_{\rho}^{(a)}$$
$$+ \frac{\delta \underset{\sim}{L}}{\delta \psi_{A}} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B} + \underset{\sim}{Q}^{A} (I_{A}{}^{B})_{\rho}{}^{\sigma} R_{B} \tag{3.2.15}$$

(R_B is short for $R^{\alpha}{}_{\beta\gamma\delta}$, and written out explicitly the last term is

$$\underset{\sim}{Q}^{A} (I_{A}{}^{B})_{\rho}{}^{\sigma} R_{B} = \underset{\sim}{Q}_{\rho}{}^{\beta\gamma\delta} R^{\sigma}{}_{\beta\gamma\delta} - 3\underset{\sim}{Q}_{\alpha}{}^{\sigma\gamma\delta} R^{\alpha}{}_{\rho\gamma\delta} \quad . \tag{3.2.16}$$

The <u>non-gravitational field equations</u> are

$$\frac{\delta \underset{\sim}{L}}{\delta \psi_A} = 0 \quad . \tag{3.2.17}$$

(3.2.15) now gives (modulo the non-gravitational field equations (3.2.17)) the identity (3.2.12) in the form

$$\frac{\delta \underset{\sim}{L}}{\delta e_\sigma^{(a)}} e_\rho^{(a)} = \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\rho} \underset{\sim}{N}^\sigma - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\alpha} \underset{\sim}{N}^\alpha \delta^\sigma_\rho + \underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{t}_\rho{}^\sigma + \underset{\sim}{Q}^A (I_A{}^B)_\rho{}^\sigma R_B \quad . \tag{3.2.18}$$

Replacing $\frac{\delta \underset{\sim}{L}}{\delta e_\nu^{(a)}} e^{(a)\mu}$ in (3.2.10) by the above expression gives the gravitational equations in the form

$$\begin{aligned} -(8\pi)^{-1}(-g)^{1/2} G^{\mu\nu} &= \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\rho} \underset{\sim}{N}^\nu g^{\rho\mu} - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\alpha} \underset{\sim}{N}^\alpha g^{\mu\nu} + \underset{\sim}{t}^{\mu\nu} \\ &+ (\underset{\sim}{U}^{\mu[\nu\tau]} + \underset{\sim}{U}^{\nu[\mu\tau]} + \underset{\sim}{U}^{\tau[\nu\mu]})_{|\tau} \\ &+ 4\underset{\sim}{Q}^{\tau(\mu\nu)\gamma}{}_{|\gamma\tau} + \underset{\sim}{Q}^A (I_A{}^B)^{\mu\nu} R_B \quad . \end{aligned} \tag{3.2.19}$$

§3.3 <u>Lagrangian Decomposition</u>.

We now write $\underset{\sim}{L}$ of (3.2.14) as a sum $\underset{\sim}{L} = \underset{\sim}{L}_1 + \underset{\sim}{L}_2$ where $\underset{\sim}{L}_2$ is the free field Lagrangian for ψ_A and $\underset{\sim}{L}_1$ represents the matter and its interaction with the applied fields.

$$\underset{\sim}{L} = \underset{\sim}{L}_1(\underset{\sim}{N}^\mu, e_\alpha^{(a)}, \dot{e}_\alpha^{(a)}, \underset{\sim}{\psi_A}) + \underset{\sim}{L}_2(\psi_A, \psi_{A|\alpha}, g_{\alpha\beta}) \tag{3.3.1}$$

$$\dot{e}^{(a)}_{\alpha} \equiv e^{(a)}_{\alpha|\mu}\, u^{\mu} \quad , \quad u^{\mu}u_{\mu} = -1 \ , \ \psi_{\underset{\sim}{A}} \equiv \{\psi_A, \psi_{A|\alpha}, R^{\alpha}{}_{\beta\gamma\delta}\} \quad . \qquad (3.3.2)$$

$\underset{\sim}{L}_1$ is assumed to be homogeneous of degree one in its first argument $\underset{\sim}{N}^{\mu} = n\sqrt{-g}\, u^{\mu}$ so that

$$\underset{\sim}{L}_1 = n\sqrt{-g}\, L_1(u^{\mu}, e^{(a)}_{\alpha}, \dot{e}^{(a)}_{\alpha}, \psi_{\underset{\sim}{A}}) \qquad (3.3.3)$$

where $L_1(u^{\mu},\ldots)$ is obtained from $\underset{\sim}{L}_1(\underset{\sim}{N}^{\mu},\ldots)$ by simply replacing $\underset{\sim}{N}^{\mu}$ with u^{μ}. This assumption will be clearly satisfied since for dustlike material the flux of material four momentum, and of spin, (defined as derivatives of $\underset{\sim}{L}_1(\underset{\sim}{N}^{\mu},\ldots)$) will be proportional to the particle density n .

L_1 is a single particle Lagrangian as in (2.3.1) with parameter λ set equal to proper time s . We have

$$\underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta} \equiv \partial\underset{\sim}{L}/\partial R^{\alpha}{}_{\beta\gamma\delta} = n\sqrt{-g}\ \partial L_1/\partial R^{\alpha}{}_{\beta\gamma\delta} = n\sqrt{-g}\ Q_{\alpha}{}^{\beta\gamma\delta} \qquad (3.3.4)$$

where $Q_{\alpha}{}^{\beta\gamma\delta}$ is the quadrupole moment per particle. Define a "spin flux"

$$\underset{\sim}{S}^{\mu\tau\nu} = 2\ \underset{\sim}{U}^{\nu[\mu\tau]} \quad . \qquad (3.3.5)$$

Then, with $\Phi_{\underset{\sim}{A}|\alpha} = \{\psi_{A|\alpha}, e^{(a)}_{\alpha|\beta}\}$, we have

$$\underset{\sim}{S}^{\mu\tau\nu} = 2\ \underset{\sim}{U}^{\nu[\mu\tau]} = 2(\partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}|\nu})(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\tau\mu]}\ \Phi_{\underset{\sim}{B}}$$

$$= 2(\partial\underset{\sim}{L}/\partial\psi_{A|\nu})(I_A{}^B)^{[\tau\mu]}\psi_B + 2(\partial\underset{\sim}{L}/\partial e^{(a)}_{\alpha|\nu})(-\delta^{[\mu}_{\alpha}g^{\beta\tau]})e^{(a)}_{\beta}$$

$$= \underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} - 2(\partial\underset{\sim}{L}_1/\partial e^{(a)}_{[\mu|\nu})\ e^{(a)\tau]}$$

(defining $\underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} \equiv 2\, \underset{\sim}{U}^{\nu[\mu\tau]}_{(\psi)} \equiv 2(\partial\underset{\sim}{L}/\partial\psi_{A|\nu})(I_A{}^B)^{[\tau\mu]}\, \psi_B$, the spin flux of the field).

The spin flux therefore consists of a field and a matter part,

$$\underset{\sim}{S}^{\mu\tau\nu} = \underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} + \underset{\sim}{S}^{\mu\tau\nu}_{(mat)} \quad , \qquad (3.3.6)$$

where

$$\underset{\sim}{S}^{\mu\tau\nu}_{(mat)} = -2(\partial\underset{\sim}{L}_1/\partial e^{(a)}_{[\mu|\nu})e^{(a)\tau]}$$

$$= -2\sqrt{-g}\; n(\partial L_1/\partial e^{(a)}_{[\mu|\nu})e^{(a)\tau]} = -2\sqrt{-g}\; n(\partial L_1/\partial \dot{e}^{(a)}_{[\mu})u^{\nu}\; e^{(a)\tau]}$$

$$= -\sqrt{-g}\; n\; S^{\tau\mu}\; u^{\nu} = n\sqrt{-g}\; S^{\mu\tau}\; u^{\nu} \qquad (3.3.7)$$

$S^{\mu\tau}$ being the spin angular momentum per particle (see definition (2.3.4) with $\lambda = s$). Equations (3.3.4), (3.3.5) and (3.3.6) describe the last three terms in the gravitational equations (3.2.19). All that remains is to examine the content of the other terms forming the right hand side of (3.2.19).

First, noting that $\underset{\sim}{L}$ is a function of $e^{(a)}_{\alpha|\beta}$ (see (3.2.14)) whilst $\underset{\sim}{L}_1$ is expressed as a function of $\dot{e}^{(a)}_{\alpha}$, since

$$\dot{e}^{(a)}_{\alpha} = e^{(a)}_{\alpha|\beta}\; u^{\beta} = e^{(a)}_{\alpha|\beta}\; \underset{\sim}{N}^{\beta} \,/\, (-\underset{\sim}{N}_{\lambda}\underset{\sim}{N}^{\lambda})^{1/2} \quad ,$$

we have

$$\frac{\partial\underset{\sim}{L}}{\partial\underset{\sim}{N}^{\rho}} = \frac{\partial\underset{\sim}{L}_1}{\partial\underset{\sim}{N}^{\rho}} + \frac{\partial\underset{\sim}{L}_1}{\partial\dot{e}^{(a)}_{\alpha}}\; e^{(a)}_{\alpha|\beta}\; (-\underset{\sim}{N}_{\lambda}\underset{\sim}{N}^{\lambda})^{-1/2}(\delta^{\beta}_{\rho} + u^{\beta}u_{\rho}) \qquad (3.3.8)$$

$$= \frac{\partial L_1}{\partial u^\rho} + \frac{\partial L_1}{\partial \dot{e}^{(a)}_\alpha}\; e^{(a)}_{\alpha|\beta} \quad (\delta^\beta_\rho + u^\beta u_\rho) \tag{3.3.9}$$

(by the homogeneity of $\underset{\sim}{L}_1 = (-\underset{\sim}{N}_\lambda \underset{\sim}{N}^\lambda)^{1/2} L_1$)

$$\therefore \qquad \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\alpha} \underset{\sim}{N}^\alpha = \frac{\partial \underset{\sim}{L}_1}{\partial \underset{\sim}{N}^\alpha} \underset{\sim}{N}^\alpha = \underset{\sim}{L}_1 \tag{3.3.10}$$

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\rho} \underset{\sim}{N}^\nu g^{\rho\mu} = \frac{\partial L_1}{\partial u^\rho} \underset{\sim}{N}^\nu g^{\rho\mu} + \frac{\partial L_1}{\partial \dot{e}^{(a)}_\alpha} e^{(a)|\mu}_\alpha \underset{\sim}{N}^\nu + \frac{\partial L_1}{\partial \dot{e}^{(a)}_\alpha} \dot{e}^{(a)}_\alpha u^\mu \underset{\sim}{N}^\nu$$

$$= P^\mu \underset{\sim}{N}^\nu + \frac{\partial L_1}{\partial \dot{e}^{(a)}_\alpha} e^{(a)|\mu}_\alpha \underset{\sim}{N}^\nu \tag{3.3.11}$$

where P^μ is the canonical momentum per particle defined by (2.3.5) (with λ set equal to s). $\underset{\sim}{t}^{\mu\nu}$ defined by (3.2.13) is

$$\underset{\sim}{t}^{\mu\nu} \equiv \underset{\sim}{L}\, g^{\mu\nu} - \Phi_{\underset{\sim}{A}}{}^{|\mu}\, \partial\underset{\sim}{L}/\partial\Phi_{\underset{\sim}{A}|\nu}$$

$$= (\underset{\sim}{L}_1 + \underset{\sim}{L}_2) g^{\mu\nu} - \psi_A{}^{|\mu}\, \partial\underset{\sim}{L}/\partial\psi_{A|\nu} - e^{(a)|\mu}_\alpha\, \partial\underset{\sim}{L}/\partial e^{(a)}_{\alpha|\nu}$$

$$= \underset{\sim}{L}_1\, g^{\mu\nu} + (-g)^{1/2}\, t^{\mu\nu}_{(\psi)} - e^{(a)|\mu}_\alpha \frac{\partial L_1}{\partial \dot{e}^{(a)}_\alpha} \underset{\sim}{N}^\nu \tag{3.3.12}$$

where the canonical energy tensor for the fields ψ_A is defined by

$$(-g)^{1/2}\, t^{\mu\nu}_{(\psi)} = \underset{\sim}{L}_2\, g^{\mu\nu} - \psi_A{}^{|\mu}\, \partial\underset{\sim}{L}/\partial\psi_{A|\nu} \quad . \tag{3.3.13}$$

Equations (3.3.10, 11, 12) together give

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\rho} \underset{\sim}{N}^\nu g^{\rho\mu} - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^\alpha} \underset{\sim}{N}^\alpha g^{\mu\nu} + \underset{\sim}{t}^{\mu\nu} = P^\mu \underset{\sim}{N}^\nu + (-g)^{1/2}\, t^{\mu\nu}_{(\psi)} \quad . \tag{3.3.14}$$

Substitution of (3.3.4), (3.3.5) and (3.3.14) into (3.2.19) gives the final form of the gravitational field equations for polarised dustlike material:

$$\boxed{-(8\pi)^{-1}\, G^{\mu\nu} = T^{\mu\nu} \equiv n\, P^{\mu} u^{\nu} + t^{\mu\nu}_{(\psi)} + \tfrac{1}{2}(-g)^{-1/2}(\underset{\sim}{S}^{\mu\tau\nu} + \underset{\sim}{S}^{\nu\tau\mu} + \underset{\sim}{S}^{\nu\mu\tau})_{|\tau} + 4(n\, Q^{\tau(\mu\nu)\gamma})_{|\gamma\tau} + n\, Q^{A}(I_{A}{}^{B})^{\mu\nu}\, R_{B}} \qquad (3.3.15)$$

$T^{\mu\nu}$, coupled above to the Einstein tensor $G^{\mu\nu}$, is the symmetric, covariantly constant, total energy momentum tensor for material and fields. It is written as a sum of a material energy momentum tensor (i.e. a momentum flux) $n\, P^{\mu} u^{\nu}$, a canonical energy tensor $t^{\mu\nu}_{(\psi)}$ for the fields ψ_A defined by (3.3.13), a spin term where the spin flux $\underset{\sim}{S}^{\mu\tau\nu}$ is a sum of a field and a matter part $\underset{\sim}{S}^{\mu\tau\nu} = \underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} + (-g)^{1/2}\, n\, S^{\mu\tau}\, u^{\nu}$ (see 3.3.6, 7) and a contribution from gravitational quadrupole moments.

§3.4 Single Particle Equations From the Four Dimensional Action Integral.

It is easy to show that the spin equations of motion (3.2.11), i.e. $e^{(a)[\mu} \dfrac{\delta \underset{\sim}{L}}{\delta e^{(a)}_{\nu]}} = 0$, are equivalent to the single particle spin equations (2.3.6). First, since $\underset{\sim}{L}_2$ depends on $e^{(a)}_{\nu}$ only via the metric,

$$\frac{\delta \underset{\sim}{L}_2}{\delta e^{(a)}_{\nu}} = \frac{\partial \underset{\sim}{L}_2}{\partial e^{(a)}_{\nu}} = \frac{\partial \underset{\sim}{L}_2}{\partial g_{\alpha\beta}}\, 2e_{(a)(\alpha}\delta^{\nu}{}_{\beta)} = \frac{\partial \underset{\sim}{L}_2}{\partial g_{\alpha\nu}}\, 2e_{(a)\alpha}$$

$$\therefore \quad \frac{\delta \underset{\sim}{L}_2}{\delta e^{(a)}_{\nu}}\, e^{(a)\mu} = 2\, \frac{\partial \underset{\sim}{L}_2}{\partial g_{\mu\nu}}$$

which is symmetric in μ,ν . The spin equations are therefore

$$e^{(a)[\mu} \frac{\delta \underset{\sim}{L}_1}{\delta e^{(a)}_{\nu]}} = 0 \quad .$$

$$\frac{\delta \underset{\sim}{L}_1}{\delta e_\nu^{(a)}} = \frac{\partial \underset{\sim}{L}_1}{\partial e_\nu^{(a)}} - \left(\frac{\partial \underset{\sim}{L}_1}{\partial e^{(a)}_{\nu|\lambda}}\right)_{|\lambda} = \frac{\partial}{\partial e_\nu^{(a)}} (n\sqrt{-g}\, L_1) - \left(n\sqrt{-g}\, \frac{\partial L_1}{\partial \dot{e}_\nu^{(a)}}\, u^\lambda\right)_{|\lambda}$$

$$= \frac{\partial \sqrt{-g}}{\partial e_\nu^{(a)}}\, n\, L_1 + n\sqrt{-g}\, \frac{\partial L_1}{\partial e_\nu^{(a)}} - n\sqrt{-g}\, \frac{\delta}{\delta s} \left(\frac{\partial L_1}{\partial \dot{e}_\nu^{(a)}}\right) - \frac{\partial L_1}{\partial \dot{e}_\nu^{(a)}} (\sqrt{-g}\, n u^\lambda)_{|\lambda}$$

i.e.,

$$\frac{\delta \underset{\sim}{L}_1}{\delta e_\nu^{(a)}} = n\sqrt{-g} \left(\frac{\partial L_1}{\partial e_\nu^{(a)}} - \frac{\delta}{\delta s} \left(\frac{\partial L_1}{\partial \dot{e}_\nu^{(a)}}\right)\right) - \frac{\partial L_1}{\partial \dot{e}_\nu^{(a)}} (\sqrt{-g}\, n\, u^\lambda)_{|\lambda} + \frac{\partial \sqrt{-g}}{\partial e_\nu^{(a)}}\, n \quad L_1 \quad .$$

The last term gives a term symmetric in μ and ν and disappearing on antisymmetrisation. So, assuming that the number of particles is conserved, i.e. that $\underset{\sim}{N}^\mu{}_{|\mu} = (\sqrt{-g}\, n\, u^\mu)_{|\mu} = 0$, then it follows that (3.2.11) and (2.3.6) are equivalent.

Since the Einstein tensor satisfies the contracted Bianchi identity $G^{\mu\nu}{}_{|\nu} = 0$, the total energy tensor must also satisfy $T^{\mu\nu}{}_{|\nu} = 0$ if the gravitational field equations (3.3.15) are to be consistent. One may simplify equations $T^{\mu\nu}{}_{|\nu} = 0$ using the spin equations and non-gravitational field equations to obtain equations of motion for P^μ (cf. the derivation of the geodesic equation for "pure" dust from $T^{\mu\nu}{}_{|\nu} = 0$ with $T^{\mu\nu} = \rho\, u^\mu\, u^\nu$) .

However, a much easier approach is to "vary the world lines" in the action integral, the problem reducing to a variation of a one parameter action integral of the single particle type. The integral curves of $\underset{\sim}{N}^\mu$,

given by equations $\partial x^{\mu}(a^{i},\lambda)/\partial\lambda = v^{\mu} = u^{\mu}\, ds/d\lambda$, define the particle world lines $x^{\mu} = x^{\mu}(a^{i},\lambda)$ where a^{i} , $i = 1,2,3$, are constant along the world line $(u^{\mu}\,\partial_{\mu}\, a^{i} = 0)$ and λ is an arbitrary parameter along the curves.

$$\sqrt{-g}\; d^{4}x = \sqrt{-g}\; \left|\frac{\partial x}{\partial(\lambda,a)}\right|\; d^{3}a\; d\lambda$$

$$= \sqrt{-g}\; \epsilon_{\mu\alpha\beta\gamma}\; v^{\mu}\; E^{\alpha}_{(1)}\; E^{\beta}_{(2)}\; E^{\gamma}_{(3)}\; d^{3}a\; d\lambda \qquad (3.4.1)$$

$(E^{\alpha}_{(i)} \equiv \partial x^{\alpha}/\partial a^{i}$, $v^{\mu} = \partial x^{\mu}/\partial\lambda$, $\epsilon_{\mu\alpha\beta\gamma}$ is the Levi-Civita permutation symbol.) The $E^{\alpha}_{(i)}$ are tangential to the λ = constant surface. If n_{μ} is the unit normal to this surface and $\sqrt{{}^{3}g}$ is the square root of the surface three-metric then

$$\sqrt{-g}\; \epsilon_{\mu\alpha\beta\gamma}\; E^{\alpha}_{(1)}\; E^{\beta}_{(2)}\; E^{\gamma}_{(3)} = -\sqrt{{}^{3}g}\;\; n_{\mu}$$

$$\sqrt{-g}\; d^{4}x = -n_{\mu}\; v^{\mu}\, \sqrt{{}^{3}g}\; d^{3}a\; d\lambda = -n_{\mu}\; v^{\mu}\; d\Sigma\; d\lambda \qquad (3.4.2)$$

where $d\Sigma = \sqrt{{}^{3}g}\; d^{3}a$ is the 3-area of section λ = constant of the infinitesimal tube $\{a^{i},d^{3}a\}$. The numerical flux across $d\Sigma$, $-n\, u^{\mu}\, n_{\mu}\, d\Sigma$, is constant along the tube since $(n\, u^{\mu})_{|\mu} = 0$. Denoting this, the number of particles in the tube $d^{3}a$, by $N(a^{i})\, d^{3}a$,

$$N(a^{i})d^{3}a \equiv -n\; u^{\mu}\; n_{\mu}\; d\Sigma\;\; , \qquad (3.4.3)$$

then by (3.4.2), (3.4.3), and the homogeniety of $\underset{\sim}{L}_{1}$,

$$\underset{\sim}{L}_{1}(\underset{\sim}{N}^{\alpha},\ldots)d^{4}x = L_{1}(u^{\alpha},\ldots)\; n\sqrt{-g}\; d^{4}x$$

$$= L_1(u^\alpha,\ldots)n(-n_\mu v^\mu)d\Sigma\ d\lambda = L_1(v^\alpha,\ldots)n(-n_\mu u^\mu)\ d\Sigma\ d\lambda$$

$$= L_1(v^\alpha,\ldots)\ N\ d^3a\ d\lambda\ .$$

$$\int \underset{\approx}{L}_1\ d^4x = \int N(a^i)d^3a \int L_1(v^\mu,\ldots)\ d\lambda\ . \qquad (3.4.4)$$

Keeping the applied fields ψ_A , $g_{\mu\nu}$ fixed, we extremise the action integral I for arbitrary variations in the flux field $\underset{\sim}{N}^\mu$ (a variation in $\underset{\sim}{N}^\mu$ inducing a corresponding variation in the integral curves of $\underset{\sim}{N}^\mu$) subject to $\delta e_\alpha^{(a)} = 0$ and $dN = 0$. The condition that N is constant simply requires that the number of particles $N\ d^3a$ in a tube d^3a remains fixed on variation. Since $\underset{\approx}{L}_2$ and R are not varied we have by (3.4.4)

$$dI = \int N(a^i)d^3a \int d\ L_1(v^\mu,\ldots)\ d\lambda \qquad (3.4.5)$$

and resultant equations are exactly as in Chapter Two, §2.3.

§3.5 Special Case: Maxwell Einstein Field.

Specialising the results of this chapter to the media in interaction with an electromagnetic field, we take the fields ψ_A to be the four vector potential A_α .

Making the same assumptions for the dependence of $\underset{\sim}{L}$ upon A_α as in §2.4 for the particle Lagrangian, we take $\underset{\sim}{L}$ to depend on A_α only through a term $e\ A_\alpha\ \underset{\sim}{N}^\alpha$ and to depend upon $A_{\alpha|\beta}$ only via $F_{\beta\alpha} = A_{\alpha|\beta} - A_{\beta|\alpha}$.

We have at once the <u>electromagnetic field equations</u>

$$0 = \frac{\delta \underset{\sim}{L}}{\delta A_\alpha} = \frac{\partial \underset{\sim}{L}}{\partial A_\alpha} - \left(\frac{\partial \underset{\sim}{L}}{\partial A_{\alpha|\beta}}\right)_{|\beta}$$

i.e.,

$$\boxed{H^{\alpha\beta}{}_{|\beta} = 4\pi\ J^\alpha} \tag{3.5.1}$$

defining current J^α and antisymmetric tensor $H^{\alpha\beta}$ by

$$\sqrt{-g}\ J^\alpha = \frac{\partial \underset{\sim}{L}}{\partial A_\alpha} = e\ \underset{\sim}{N}^\alpha = \sqrt{-g}\ e\ n\ u^\alpha \tag{3.5.2}$$

$$\sqrt{-g}\ H^{\alpha\beta} = -8\pi \frac{\partial \underset{\sim}{L}}{\partial F_{\alpha\beta}} = 4\pi \frac{\partial \underset{\sim}{L}}{\partial A_{\alpha|\beta}} \quad . \tag{3.5.3}$$

Antisymmetry of $H^{\alpha\beta}$ and equations (3.5.1) imply the conservation equation $J^\alpha{}_{|\alpha} = 0$ for charge, and $J^\alpha = e\ n\ u^\alpha$, $J^\alpha{}_{|\alpha} = 0$ and $(n\ u^\alpha)_{|\alpha} = 0$ together imply $de/ds = 0$. The canonical energy tensor $t^{\mu\nu}_{(\psi)}$ for the field, given by (3.3.13), is

$$t^{\mu\nu}_{(\psi)} = (-g)^{-1/2}\ \underset{\approx}{L}_2\ g^{\mu\nu} - (4\pi)^{-1}\ A_\alpha{}^{|\mu}\ H^{\alpha\nu} \tag{3.5.4}$$

and the field spin flux (see definition at top of page 28) is

$$\begin{aligned} \underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} &= 2\sqrt{-g}\ (4\pi)^{-1}\ H^{\alpha\nu}\ (-g^{\beta[\tau}\delta^{\mu]}_\alpha)\ A_\beta \\ &= \sqrt{-g}\ (2\pi)^{-1}\ H^{\nu[\mu}A^{\tau]} = \sqrt{-g}\ (2\pi)^{-1}\ A^{[\mu}H^{\tau]\nu} \quad . \end{aligned} \tag{3.5.5}$$

This implies

$$\underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} + \underset{\sim}{S}^{\nu\tau\mu}_{(\psi)} + \underset{\sim}{S}^{\nu\mu\tau}_{(\psi)} = \sqrt{-g}\ (2\pi)^{-1}\ A^{\mu}\ H^{\tau\nu} \quad . \tag{3.5.6}$$

(3.5.4) and (3.5.6) together give, using (3.5.1),

$$t^{\mu\nu}_{(\psi)} + \tfrac{1}{2}(-g)^{-1/2}\ (\underset{\sim}{S}^{\mu\tau\nu}_{(\psi)} + \underset{\sim}{S}^{\nu\tau\mu}_{(\psi)} + \underset{\sim}{S}^{\nu\mu\tau}_{(\psi)})_{|\tau}$$

$$= -A^{\mu}\ J^{\nu} + (4\pi)^{-1}\ F_{\tau}{}^{\mu}\ H^{\tau\nu} + (-g)^{-1/2}\ \underset{\sim}{L}_2\ g^{\mu\nu} \quad . \tag{3.5.7}$$

Now defining a new, gauge invariant, "kinetic" momentum p_{μ} as in §2.4, $p_{\mu} = P_{\mu} - e\ A_{\mu}$. Matter tensor $T^{\mu\nu}_{(mat)} \equiv n\ p^{\mu}\ u^{\nu}$ is then

$$T^{\mu\nu}_{(mat)} = n\ P^{\mu}\ u^{\nu} - A^{\mu}\ J^{\nu} \quad . \tag{3.5.8}$$

Define a gauge invariant electromagnetic tensor by

$$T^{\mu\nu}_{(e.m.)} \equiv (4\pi)^{-1}\ F_{\tau}{}^{\mu}\ H^{\tau\nu} + (-g)^{-1/2}\ \underset{\sim}{L}_2\ g^{\mu\nu} \quad . \tag{3.5.9}$$

Equations (3.5.7), (3.5.8) and (3.5.9) now give the final form of the <u>gravitational field equations</u> (3.3.15) as

$$\boxed{\begin{aligned} -(8\pi)^{-1}\ G^{\mu\nu} = T^{\mu\nu} \equiv T^{\mu\nu}_{(mat)} + T^{\mu\nu}_{(e.m.)} + \tfrac{1}{2}(S^{\mu\tau\nu}_{(mat)} + S^{\nu\tau\mu}_{(mat)} + S^{\nu\mu\tau}_{(mat)})_{|\tau} \\ + 4(nQ^{\tau(\mu\nu)\gamma})_{|\gamma\tau} + nQ^{A}(I_{A}{}^{B})^{\mu\nu}\ R_{B} \quad . \qquad (3.5.10) \end{aligned}}$$

$T^{\mu\nu}_{(mat)} = n\ p^{\mu}\ u^{\nu}$ and $S^{\mu\tau\nu}_{(mat)} = n\ S^{\mu\tau}\ u^{\nu}$ are, respectively, the material fluxes of momentum and spin.

If we choose for $\underset{\approx}{L}_2$ the usual free field electromagnetic Lagrangian $-(16\pi)^{-1} \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}$, and define a polarisation tensor $M^{\alpha\beta}$ by

$$\sqrt{-g}\, M^{\alpha\beta} = \partial\underset{\approx}{L}_1/\partial A_{\beta|\alpha} = 2\, \partial\underset{\approx}{L}_1/\partial F_{\alpha\beta} \quad ,$$

then (3.5.3) gives

$$H^{\alpha\beta} = F^{\alpha\beta} - 4\pi\, M^{\alpha\beta} \tag{3.5.11}$$

and

$$F^{\alpha\beta}{}_{|\beta} = 4\pi\, (J^{\alpha} + M^{\alpha\beta}{}_{|\beta}) \tag{3.5.12}$$

are the electromagnetic field equations.

CHAPTER FOUR

HIGHER DERIVATIVE COUPLING

§4.1 Generalisation of the Rosenfeld-Belinfante Identity (1.14).

In Chapter Three we considered a Lagrangian $\underset{\sim}{L}$ depending only on variables $\Phi_{\underset{\sim}{A}}$ and first covariant derivatives $\Phi_{\underset{\sim}{A}|\alpha}$. We now discuss the more general situation where $\underset{\sim}{L}$ is allowed to depend on higher derivatives of $\Phi_{\underset{\sim}{A}}$.

Since any derivative $\Phi_{\underset{\sim}{A}|\alpha_1 \ldots \alpha_n}$ can be rewritten, making use of the Ricci commutation relations (1.8), in terms of the symmetrised derivatives $\Phi_{\underset{\sim}{A}}$, $\Phi_{\underset{\sim}{A}|\alpha}, \ldots \Phi_{\underset{\sim}{A}|(\alpha_1 \ldots \alpha_n)}$, $R^{\alpha}{}_{\beta\gamma\delta}$, $\ldots$ $R^{\alpha}{}_{\beta\gamma\delta|(\alpha_1 \ldots \alpha_n)}$ we can assume that our Lagrangian is dependent only on symmetrised derivatives, $\underset{\sim}{L} = \underset{\sim}{L}(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)})$, $n = 0,1,2,\ldots$, $\underset{\sim}{\alpha}(0)$ denoting the empty set and $\underset{\sim}{\alpha}(n)$, $n \geq 1$, denoting a symmetrised set of indices $\underset{\sim}{\alpha}(n) = (\alpha_1 \ldots \alpha_n)$. A repeated set of indices $\underset{\sim}{\alpha}(n)$ will, in the usual way, imply summation over the respective indices. The use of symmetrised sets enables one to deal with indices without any worry about their ordering.

The first step is to generalise the identity (1.14), the condition that $\underset{\sim}{L}$ be a scalar density. We consider a scalar density

$$\underset{\sim}{L} = \underset{\sim}{L}(\Phi_{A|\underset{\sim}{\alpha}(n)}) \quad , \quad n = 0,1,2,\ldots \tag{4.1.1}$$

and make the following definitions:

$$\underset{\sim}{L}^{A} \equiv \partial\underset{\sim}{L}/\partial\Phi_{A} \ , \ \underset{\sim}{L}^{A\alpha_1\ldots\alpha_n} = \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} \equiv \partial\underset{\sim}{L}/\partial\Phi_{A|(\alpha_1\ldots\alpha_n)} \tag{4.1.2}$$

$$\underset{\sim}{L}_{*}^{A\alpha_1\ldots\alpha_n} = \underset{\sim}{L}_{*}^{A\underset{\sim}{\alpha}(n)} \equiv \sum_{m=0}^{\infty} (-1)^m \, \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)\underset{\sim}{\beta}(m)}{}_{|\underset{\sim}{\beta}(m)} \tag{4.1.3}$$

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho} \equiv (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} \underset{\sim}{L}_{*}^{A\tau\underset{\sim}{\alpha}(n)} \, \Phi_{B|\underset{\sim}{\alpha}(n)} \tag{4.1.4}$$

$$\underset{\sim}{t}_{\rho}{}^{\sigma} \equiv \delta^{\sigma}_{\rho} \underset{\sim}{L} - \sum_{n=0}^{\infty} (n+1) \, \Phi_{A|(\rho\underset{\sim}{\alpha}(n))} \, \underset{\sim}{L}^{A\sigma\underset{\sim}{\alpha}(n)} \quad . \tag{4.1.5}$$

The condition (1.10) that $\underset{\sim}{L}(\Phi_{\underset{\sim}{A}})$ be a scalar density is

$$\frac{\partial\underset{\sim}{L}}{\partial\Phi_{\underset{\sim}{A}}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma \, \Phi_{\underset{\sim}{B}} + \delta^{\sigma}_{\rho} \underset{\sim}{L} = 0 \quad . \tag{4.1.6}$$

For a Lagrangian (4.1.1), $\Phi_{\underset{\sim}{A}} = \{\Phi_{A|\underset{\sim}{\alpha}(n)} \ , \ n = 0,1,2,\ldots\}$, this is

$$\sum_{n=0}^{\infty} \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} \, (I_{A\underset{\sim}{\alpha}(n)}{}^{B\underset{\sim}{\beta}(n)})_\rho{}^\sigma \, \Phi_{B|\underset{\sim}{\beta}(n)} + \delta^{\sigma}_{\rho} \underset{\sim}{L} = 0 \quad . \tag{4.1.7}$$

Repeated use of (1.5) gives

$$\begin{aligned} \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} \, (I_{A\underset{\sim}{\alpha}(n)}{}^{B\underset{\sim}{\beta}(n)})_\rho{}^\sigma \, \Phi_{B|\underset{\sim}{\beta}(n)} &= \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} (I_A{}^B)_\rho{}^\sigma \, \Phi_{B|\underset{\sim}{\alpha}(n)} \\ &\quad -n \, \underset{\sim}{L}^{A\sigma\underset{\sim}{\alpha}(n-1)} \, \Phi_{A|(\rho\underset{\sim}{\alpha}(n-1))} \ , \quad n = 1,2,\ldots \end{aligned} \tag{4.1.8}$$

Substitution of (4.1.8) into (4.1.7) gives identity (4.1.7) in the form

$$\sum_{n=0}^{\infty} \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} \, (I_A{}^B)_\rho{}^\sigma \, \Phi_{B|\underset{\sim}{\alpha}(n)} + \underset{\sim}{t}_{\rho}{}^{\sigma} = 0 \quad . \tag{4.1.9}$$

From (4.1.3),

$$\underset{\sim}{L}_*^{A\tau\underset{\sim}{\alpha}(n)}{}_{|\tau} = \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} - \underset{\sim}{L}_*^{A\underset{\sim}{\alpha}(n)} \quad . \tag{4.1.10}$$

Therefore the divergence of (4.1.4) gives

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} = (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} ((\underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)} - \underset{\sim}{L}_*^{A\underset{\sim}{\alpha}(n)})\Phi_{B|\underset{\sim}{\alpha}(n)} + \underset{\sim}{L}_*^{A\tau\underset{\sim}{\alpha}(n)}\Phi_{B|\tau\underset{\sim}{\alpha}(n)})$$

i.e.,

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} = (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} \underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)}\Phi_{B|\underset{\sim}{\alpha}(n)} - (I_A{}^B)_\rho{}^\sigma \underset{\sim}{L}_*^A \Phi_B \quad . \tag{4.1.11}$$

Finally, the insertion of the above into (4.1.9) gives the generalised form of identity (1.14):

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{L}_*^A (I_A{}^B)_\rho{}^\sigma \Phi_B + \underset{\sim}{t}_\rho{}^\sigma = 0 \quad . \tag{4.1.12}$$

§4.2 Tetrad Variation.

The action integral is now

$$I = \int (-(16\pi)^{-1}(-g)^{1/2} R + \underset{\sim}{L}(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)}))\, d^4x \tag{4.2.1}$$

with $\Phi_{\underset{\sim}{A}} = \{\underset{\sim}{N}^\alpha, e_\alpha^{(a)}, \psi_A, R^\alpha{}_{\beta\gamma\delta}\}$. Symmetrised derivatives of any order are now allowed to appear in $\underset{\sim}{L}$, but we assume again that no derivatives of $\underset{\sim}{N}^\alpha$ occur and only first derivatives of $e_\alpha^{(a)}$ are allowed. i.e.,

$$\Phi_{\underset{\sim}{A}|\alpha} = \Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(1)} = \{e^{(a)}_{\alpha|\beta}, \psi_{A|\beta}, R^{\alpha}{}_{\beta\gamma\delta|\rho}\} \tag{4.2.2}$$

and

$$\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)} = \{\psi_{A|\underset{\sim}{\alpha}(n)}, R^{\alpha}{}_{\beta\gamma\delta|\underset{\sim}{\alpha}(n)}\} \quad , \quad n \geq 2 \quad .$$

For a variation $de^{(a)}_{\alpha}$ in the tetrad field, the change in $\underset{\sim}{L}$ is

$$d\underset{\sim}{L} = \sum_{n=0}^{\infty} \underset{\sim}{L}^{\underset{\sim}{A}\underset{\sim}{\alpha}(n)} \, d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)}) \quad . \tag{4.2.3}$$

From (1.7) the variation $d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)})$ for $n \geq 1$ is given by

$$d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)}) = (d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n-1)}))|\alpha_n + (I_{\underset{\sim}{A}\underset{\sim}{\alpha}(n-1)}{}^{\underset{\sim}{B}\underset{\sim}{\beta}(n-1)})_{\rho}{}^{\sigma} \Phi_{\underset{\sim}{B}|\underset{\sim}{\beta}(n-1)} \, d\,\Gamma^{\rho}_{\sigma\alpha_n}$$

(where α_n here is "symmetrised" along with the other $\underset{\sim}{\alpha}(n-1)$). (4.1.8) then gives (for $n \geq 1$)

$$\begin{aligned} \underset{\sim}{L}^{\underset{\sim}{A}\underset{\sim}{\alpha}(n)} \, d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)}) &= \underset{\sim}{L}^{\underset{\sim}{A}\underset{\sim}{\alpha}(n)} (d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n-1)}))|\alpha_n \\ &+ \underset{\sim}{L}^{\underset{\sim}{A}\underset{\sim}{\alpha}(n)} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \, \Phi_{\underset{\sim}{B}|\underset{\sim}{\alpha}(n-1)} \, d\,\Gamma^{\rho}_{\sigma\alpha_n} \\ &-(n-1)\underset{\sim}{L}^{\underset{\sim}{A}\sigma\underset{\sim}{\alpha}(n-2)\alpha_n} \, \Phi_{\underset{\sim}{A}|(\rho\underset{\sim}{\alpha}(n-2))} \, d\,\Gamma^{\rho}_{\sigma\alpha_n} \quad . \end{aligned} \tag{4.2.4}$$

Substitution of the above into (4.2.3) and an "integration by parts" gives

$$d\underset{\sim}{L} = \underset{\sim}{L}^{\underset{\sim}{A}} \, d\Phi_{\underset{\sim}{A}} + (\mathrm{div}) - \sum_{n=0}^{\infty} \underset{\sim}{L}^{\underset{\sim}{A}\beta_1\underset{\sim}{\alpha}(n)}{}_{|\beta_1} \, d(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)})$$

$$+ (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma \sum_{n=0}^{\infty} \underset{\sim}{L}^{\underset{\sim}{A}\tau\underset{\sim}{\alpha}(n)} \Phi_{\underset{\sim}{B}|\underset{\sim}{\alpha}(n)} \, d\,\Gamma^\rho_{\sigma\tau}$$

$$- \sum_{n=0}^{\infty} (n+1) \, \Phi_{\underset{\sim}{A}|(\rho\underset{\sim}{\alpha}(n))} \underset{\sim}{L}^{\underset{\sim}{A}\sigma\tau\underset{\sim}{\alpha}(n)} \, d\,\Gamma^\rho_{\sigma\tau} \quad . \tag{4.2.5}$$

Repetition of the above procedure (we now have $-\underset{\sim}{L}^{\underset{\sim}{A}\beta_1\underset{\sim}{\alpha}(n)}{}_{|\beta_1}$ replacing $\underset{\sim}{L}^{\underset{\sim}{A}\underset{\sim}{\alpha}(n)}$ in (4.2.3), (4.2.4)) will give

$$d\underset{\sim}{L} = (\text{div}) + \underset{\sim *}{L}^{\underset{\sim}{A}} \, d\Phi_{\underset{\sim}{A}} + (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma \sum_{n=0}^{\infty} \underset{\sim *}{L}^{\underset{\sim}{A}\tau\underset{\sim}{\alpha}(n)} \Phi_{\underset{\sim}{B}|\underset{\sim}{\alpha}(n)} \, d\Gamma^\rho_{\sigma\tau}$$

$$- \sum_{n=0}^{\infty} (n+1) \, \Phi_{\underset{\sim}{A}|(\rho\underset{\sim}{\alpha}(n))} \underset{\sim *}{L}^{\underset{\sim}{A}\sigma\tau\underset{\sim}{\alpha}(n)} \, d\Gamma^\rho_{\sigma\tau} \quad . \tag{4.2.6}$$

Define

$$\underset{\sim}{Y}^{\tau\sigma}{}_\rho \equiv \sum_{n=0}^{\infty} (n+1)\Phi_{\underset{\sim}{A}|(\rho\underset{\sim}{\alpha}(n))} \underset{\sim *}{L}^{\underset{\sim}{A}\sigma\tau\underset{\sim}{\alpha}(n)} \quad ; \quad \underset{\sim *}{U}^{\tau\sigma}{}_\rho \equiv \underset{\sim}{U}^{\tau\sigma}{}_\rho - \underset{\sim}{Y}^{\tau\sigma}{}_\rho \tag{4.2.7}$$

then comparison with (4.1.4) yields

$$d\underset{\sim}{L} = \underset{\sim *}{L}^{\underset{\sim}{A}} \, d\Phi_{\underset{\sim}{A}} + \underset{\sim *}{U}^{\tau\sigma}{}_\rho \, d\Gamma^\rho_{\sigma\tau} + (\text{div}) \quad . \tag{4.2.8}$$

Define the gravitational multipole moments $\underset{\sim}{Q}^{A(\alpha_1\ldots\alpha_n)}$ and associated quantities $\underset{\sim *}{Q}^{A}$ as special cases of general definitions (4.1.2), (4.1.3)

$$\underset{\sim}{Q}_\lambda{}^{\beta\gamma\delta} \equiv \underset{\sim}{Q}^A \equiv \partial\underset{\sim}{L}/\partial R_A = \partial\underset{\sim}{L}/\partial R^\lambda{}_{\beta\gamma\delta} \tag{4.2.9}$$

$$\underset{\sim}{Q}^{A\underset{\sim}{\alpha}(n)} = \underset{\sim}{Q}^{A\alpha_1\ldots\alpha_n} \equiv \partial\underset{\sim}{L}/\partial R_{A|\underset{\sim}{\alpha}(n)} \tag{4.2.10}$$

$$\underset{\sim *}{Q}^{A} = \underset{\sim}{Q}_{\lambda *}^{\beta\gamma\delta} \equiv \sum_{m=0}^{\infty} (-1)^m \, \underset{\sim}{Q}^{A\underset{\sim}{\beta}(m)}{}_{|\underset{\sim}{\beta}(m)} \quad . \tag{4.2.11}$$

The set $d\Phi_{\underset{\sim}{A}}$ is $\{de_{\nu}^{(a)}, dR^{\alpha}{}_{\beta\gamma\delta}\}$, giving

$$\underset{\sim *}{L}^{\underset{\sim}{A}}\, d\Phi_{\underset{\sim}{A}} = \frac{\delta \underset{\sim}{L}}{\delta e_{\mu}^{(a)}}\, de_{\mu}^{(a)} + \underset{\sim}{Q}_{\alpha *}{}^{\beta\gamma\delta}\, dR^{\alpha}{}_{\beta\gamma\delta}$$

and therefore (4.2.8) as

$$d\underset{\sim}{L} = \frac{\delta \underset{\sim}{L}}{\delta e_{\mu}^{(a)}}\, de_{\mu}^{(a)} + \underset{\sim *}{U}^{\tau\sigma}{}_{\rho}\, d\Gamma^{\rho}{}_{\sigma\tau} + \underset{\sim}{Q}_{\alpha *}{}^{\beta\gamma\delta}\, dR^{\alpha}{}_{\beta\gamma\delta} + (\mathrm{div}) \quad . \tag{4.2.12}$$

This equation is identical to (3.2.7) except that $\underset{\sim *}{U}^{\tau\sigma}{}_{\rho}$ and $\underset{\sim}{Q}_{\alpha *}{}^{\beta\gamma\delta}$ replace $\underset{\sim}{U}^{\tau\sigma}{}_{\rho}$ and $\underset{\sim}{Q}_{\alpha}{}^{\beta\gamma\delta}$, and a calculation identical to that of Page 24 gives, as the generalisation of (3.2.10), the gravitational and spin equations:

$$\begin{aligned} -(8\pi)^{-1}(-g)^{1/2}\, G^{\mu\nu} = \frac{\delta \underset{\sim}{L}}{\delta e_{\nu}^{(a)}}\, e^{(a)\mu} &+ (\underset{\sim *}{U}^{\mu[\nu\tau]} + \underset{\sim *}{U}^{\nu[\mu\tau]} - \underset{\sim *}{U}^{\tau(\mu\nu)})_{|\tau} \\ &+ 4\underset{\sim *}{Q}^{\tau(\mu\nu)\gamma}{}_{|\gamma\tau} \quad . \end{aligned} \tag{4.2.13}$$

Since only one term is not symmetric in μ and ν , as in Chapter Three, the spin equations are

$$e^{(a)[\mu} \frac{\delta \underset{\sim}{L}}{\delta e^{(a)}_{\nu]}} = 0 \quad . \tag{4.2.14}$$

Again the Belinfante-Rosenfeld identity is used to eliminate $\frac{\delta \underset{\sim}{L}}{\delta e_{\nu}^{(a)}} \cdot e_{\mu}^{(a)}$ in (4.2.13). For

$$\underset{\sim}{L} = \underset{\sim}{L}(\Phi_{\underset{\sim}{A}|\underset{\sim}{\alpha}(n)}) \quad , \quad \Phi_{\underset{\sim}{A}} = \{\underset{\sim}{N}^{\mu}, e_{\alpha}^{(a)}, \psi_{A}, R^{\alpha}{}_{\beta\gamma\delta}\} \quad , \tag{4.2.15}$$

$$\underset{\sim}{\Phi}_{A|\underset{\sim}{\alpha}(n)} = \{\psi_{A|\underset{\sim}{\alpha}(n)}, R^{\alpha}{}_{\beta\gamma\delta|\underset{\sim}{\alpha}(n)}\} \quad , \quad n \geq 2 \quad , \tag{4.2.16}$$

we have (see (4.3.3))

$$\underset{\sim}{L}_{*}^{\underset{\sim}{A}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \underset{\sim}{\Phi}_{\underset{\sim}{B}} = \frac{\partial\underset{\sim}{L}}{\partial\underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\sigma} - \frac{\partial\underset{\sim}{L}}{\partial\underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha} \delta_{\rho}^{\sigma} - \frac{\delta\underset{\sim}{L}}{\delta e_{\sigma}{}^{(a)}} e_{\rho}{}^{(a)}$$

$$+ \underset{\sim}{L}_{*(\psi)}^{A} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B} + \underset{\sim}{Q}_{*}^{A} (I_{A}{}^{B})_{\rho}{}^{\sigma} R_{B} \tag{4.2.17}$$

where $\underset{\sim}{L}_{*(\psi)}^{A}$ denotes the special case of definition (4.1.3) with Φ_A set equal to ψ_A. i.e.

$$\underset{\sim}{L}_{*(\psi)}^{A} \equiv \sum_{m=0}^{\infty} (-1)^{m} (\partial\underset{\sim}{L}/\partial\psi_{A|\underset{\sim}{\beta}(m)})_{|\underset{\sim}{\beta}(m)} \quad . \tag{4.2.18}$$

The <u>non-gravitational</u> <u>field</u> <u>equations</u>

$$\underset{\sim}{L}_{*(\psi)}^{A} = 0 \tag{4.2.19}$$

and (4.2.17) give the Rosenfeld-Belinfante identity:

$$\underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{L}_{*}^{\underset{\sim}{A}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \underset{\sim}{\Phi}_{\underset{\sim}{B}} + \underset{\sim}{t}_{\rho}{}^{\sigma} = 0$$

in the form

$$\frac{\delta\underset{\sim}{L}}{\delta e_{\sigma}{}^{(a)}} e_{\rho}{}^{(a)} = \frac{\partial\underset{\sim}{L}}{\partial\underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\sigma} - \frac{\partial\underset{\sim}{L}}{\partial\underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha} \delta_{\rho}^{\sigma} + \underset{\sim}{U}^{\tau\sigma}{}_{\rho|\tau} + \underset{\sim}{t}_{\rho}{}^{\sigma} + \underset{\sim}{Q}_{*}^{A} (I_{A}{}^{B})_{\rho}{}^{\sigma} R_{B} \tag{4.2.20}$$

Substitution of (4.2.20) into (4.2.13) gives the gravitational equations as

$$-(8\pi)^{-1}(-g)^{1/2}\ G^{\mu\nu} = \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\nu}\ g^{\rho\mu} - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha}\ g^{\mu\nu} + \underset{\sim}{t}^{\mu\nu}$$

$$+ (\underset{\sim}{U}_*^{\mu[\nu\tau]} + \underset{\sim}{U}_*^{\nu[\mu\tau]} + \underset{\sim}{U}_*^{\tau[\nu\mu]})_{|\tau} + \underset{\sim}{Y}^{\tau\nu\mu}{}_{|\tau}$$

$$+ 4\underset{\sim}{Q}_*^{\tau(\mu\nu)\gamma}{}_{|\gamma\tau} + \underset{\sim}{Q}_*^{A}(I_A{}^B)^{\mu\nu}\ R_B \quad . \tag{4.2.21}$$

§4.3 Lagrangian Decomposition.

We continue in similar fashion to the derivation of §3.3, writing $\underset{\sim}{L}$ as a sum $\underset{\sim}{L} = \underset{\sim}{L}_1 + \underset{\sim}{L}_2$ of a free field Lagrangian $\underset{\sim}{L}_2$ for ψ_A and an $\underset{\sim}{L}_1(\underset{\sim}{N}^\mu,\ldots)$ homogeneous in $\underset{\sim}{N}^\mu$, $\underset{\sim}{L}_1(\underset{\sim}{N}^\mu,\ldots) = n\sqrt{-g}\ L_1(u^\mu,\ldots)$.

$$\underset{\sim}{L} = \underset{\sim}{L}_1(\underset{\sim}{N}^\mu, e_\alpha^{(a)}, \dot{e}_\alpha^{(a)}, \psi_{A|\underset{\sim}{\alpha}(n)}, R_{A|\underset{\sim}{\alpha}(n)}) + \underset{\sim}{L}_2(\psi_{A|\underset{\sim}{\alpha}(n)}\ g_{\alpha\beta}) \quad . \tag{4.3.1}$$

(4.2.9), (4.2.10) and (4.2.11) become

$$\underset{\sim}{Q}_\lambda{}^{\beta\gamma\delta} = \underset{\sim}{Q}^A = n\sqrt{-g}\ \partial L_1/\partial R^\lambda{}_{\beta\gamma\delta} \equiv n\sqrt{-g}\ Q_\lambda{}^{\beta\gamma\delta} \equiv n\sqrt{-g}\ Q^A \tag{4.3.2}$$

$$\underset{\sim}{Q}^{A\underset{\sim}{\alpha}(n)} = n\sqrt{-g}\ \partial L_1/\partial R_{A|\underset{\sim}{\alpha}(n)} \equiv n\sqrt{-g}\ Q^{A\underset{\sim}{\alpha}(n)} \equiv n\sqrt{-g}\ Q^{A\alpha_1\ldots\alpha_n} \tag{4.3.3}$$

$$\underset{\sim}{Q}_*^A = \underset{\sim}{Q}_{\lambda *}{}^{\beta\gamma\delta} = \sqrt{-g}\ \sum_{m=0}^{\infty} (-1)^m (n\ Q^{A\underset{\sim}{\beta}(m)})_{|\underset{\sim}{\beta}(m)} \tag{4.3.4}$$

where $Q^{A\alpha_1\ldots\alpha_n}$ are the gravitational multipole moments per particle. We continue by examining the definitions of the other terms appearing in (4.2.21), keeping in mind (4.2.15, 16) which imply:

$$\underset{\sim}{L}^{A} = \{\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\mu}} , \frac{\partial \underset{\sim}{L}}{\partial e_{\alpha}^{(a)}} , \frac{\partial \underset{\sim}{L}}{\partial \psi_{A}} \equiv \underset{\sim}{L}^{A}_{(\psi)} , \frac{\partial \underset{\sim}{L}}{\partial R^{\alpha}{}_{\beta\gamma\delta}} = \frac{\partial \underset{\sim}{L}}{\partial R_{A}} = \underset{\sim}{Q}^{A}\} \tag{4.3.5}$$

$$\underset{\sim}{L}^{A\alpha_1} = \{\frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\alpha|\alpha_1}} , \frac{\partial \underset{\sim}{L}}{\partial \psi_{A|\alpha_1}} \equiv \underset{\sim}{L}^{A\alpha_1}_{(\psi)} , \underset{\sim}{Q}^{A\alpha_1}\} \tag{4.3.6}$$

$$\underset{\sim}{L}^{A\alpha_1\ldots\alpha_n} = \{\underset{\sim}{L}^{A\alpha_1\ldots\alpha_n}_{(\psi)} \equiv \frac{\partial \underset{\sim}{L}}{\partial \psi_{A|\underset{\sim}{\alpha}(n)}} , \underset{\sim}{Q}^{A\underset{\sim}{\alpha}(n)}\} \quad (n \geq 2) \tag{4.3.7}$$

$$\underset{\sim}{L}^{A}_{*} = \{\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\mu}} , \frac{\delta \underset{\sim}{L}}{\delta e_{\alpha}^{(a)}} , \underset{\sim}{L}^{A}_{*(\psi)} , \underset{\sim}{Q}^{A}_{*}\} \tag{4.3.8}$$

$$\underset{\sim}{L}^{A\alpha_1}_{*} = \{\frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\alpha|\alpha_1}} , \underset{\sim}{L}^{A\alpha_1}_{*(\psi)} , \underset{\sim}{Q}^{A\alpha_1}_{*}\} \tag{4.3.9}$$

$$\underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)}_{*} = \{\underset{\sim}{L}^{A\underset{\sim}{\alpha}(n)}_{*(\psi)} , \underset{\sim}{Q}^{A\underset{\sim}{\alpha}(n)}_{*}\} \quad (n \geq 2) \tag{4.3.10}$$

Definition (4.2.7) and (4.3.10) gives

$$\begin{aligned} \underset{\sim}{Y}^{\tau\sigma}{}_{\rho} &= \sum_{n=0}^{\infty} (n+1)\psi_{A|(\rho\underset{\sim}{\alpha}(n))}\underset{\sim}{L}^{A\sigma\tau\underset{\sim}{\alpha}(n)}_{*(\psi)} + \sum_{n=0}^{\infty} (n+1)R_{A|(\rho\underset{\sim}{\alpha}(n))}\underset{\sim}{Q}^{A\sigma\tau\underset{\sim}{\alpha}(n)}_{*} \\ &\equiv \underset{\sim}{Y}^{\tau\sigma}_{(\psi)\rho} + \underset{\sim}{Y}^{\tau\sigma}_{(R)\rho} \quad . \end{aligned} \tag{4.3.11}$$

From (4.3.9), (4.3.10) we obtain

$$\begin{aligned} \underset{\sim}{U}^{\tau\sigma}{}_{\rho} &\equiv (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\rho}{}^{\sigma} \sum_{n=0}^{\infty} \underset{\sim}{L}^{A\tau\underset{\sim}{\alpha}(n)}_{*} \Phi_{\underset{\sim}{B}|\underset{\sim}{\alpha}(n)} \\ &= \underset{\sim}{U}^{\tau\sigma}_{(\psi)\rho} + \underset{\sim}{U}^{\tau\sigma}_{(R)\rho} + (-\delta^{\beta}_{\rho}\,\delta^{\sigma}_{\alpha}) \frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\alpha|\tau}} e^{(a)}_{\beta} \end{aligned} \tag{4.3.12}$$

$$\underset{\sim}{U}^{\tau\sigma}_{(\psi)\rho} \equiv (I_{A}{}^{B})_{\rho}{}^{\sigma} \sum_{n=0}^{\infty} \underset{\sim}{L}^{A\tau\underset{\sim}{\alpha}(n)}_{*(\psi)} \cdot \psi_{B|\underset{\sim}{\alpha}(n)} \quad ;$$

$$\underset{\sim}{U}^{\tau\sigma}_{(R)\rho} \equiv (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} \underset{\sim}{Q}_*^{A\tau\underset{\sim}{\alpha}(n)} R_{B|\underset{\sim}{\alpha}(n)} \quad . \tag{4.3.13}$$

Since

$$-\frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\sigma|\tau}} e^{(a)}_\rho = -\frac{\partial \underset{\sim}{L}_1}{\partial e^{(a)}_{\sigma|\tau}} e^{(a)}_\rho = -n\sqrt{-g}\,\frac{\partial L_1}{\partial \dot{e}^{(a)}_\sigma} e^{(a)}_\rho u^\tau$$

(4.3.12) gives

$$\underset{\sim}{U}^{\tau[\sigma\rho]} = \underset{\sim}{U}^{\tau[\sigma\rho]}_{(\psi)} + \underset{\sim}{U}^{\tau[\sigma\rho]}_{(R)} - \frac{1}{2}\underset{\sim}{N}^\tau S^{\rho\sigma} \tag{4.3.14}$$

where $S^{\rho\sigma}$ is the spin angular momentum per particle. Defining the spin flux

$$\underset{\sim}{S}_*^{\mu\tau\nu} \equiv 2\,\underset{\sim}{U}_*^{\nu[\mu\tau]} \tag{4.3.15}$$

then (4.3.14), (4.2.7) give

$$\underset{\sim}{S}_*^{\mu\tau\nu} = 2\,\underset{\sim}{U}^{\nu[\mu\tau]}_{(\psi)} + 2\,\underset{\sim}{U}^{\nu[\mu\tau]}_{(R)} - 2\,\underset{\sim}{Y}^{\nu[\mu\tau]} + S^{\mu\tau}\underset{\sim}{N}^\nu \quad . \tag{4.3.16}$$

Defining

$$\underset{\sim}{S}^{\mu\tau\nu}_{(\psi R)} \equiv 2\,\underset{\sim}{U}^{\nu[\mu\tau]}_{(\psi)} + 2\,\underset{\sim}{U}^{\nu[\mu\tau]}_{(R)} - 2\,\underset{\sim}{Y}^{\nu[\mu\tau]} \tag{4.3.17}$$

gives the spin flux in the form

$$\underset{\sim}{S}_*^{\mu\tau\nu} = \underset{\sim}{S}^{\mu\tau\nu}_{(\psi R)} + S^{\mu\tau}\underset{\sim}{N}^\nu \tag{4.3.18}$$

a sum of a matter part $S^{\mu\tau}\underset{\sim}{N}^\nu$ and a field part $\underset{\sim}{S}^{\mu\tau\nu}_{(\psi R)}$ which contains

contributions from the curvature.

Since $\underset{\sim}{L}$ is a function of $e^{(a)}_{\alpha|\beta}$ and $\underset{\sim}{L}_1$ is a function of $\dot{e}^{(a)}_{\alpha}$, a calculation identical to (3.3.8, 9, 10, 11) of Chapter Three will give

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha} = \underset{\sim}{L}_1 \tag{4.3.19}$$

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\nu} g^{\rho\mu} = P^{\mu} \underset{\sim}{N}^{\nu} + \frac{\partial \underset{\sim}{L}_1}{\partial \dot{e}^{(a)}_{\alpha}} e^{(a)|\mu}_{\alpha} \underset{\sim}{N}^{\nu} \tag{4.3.20}$$

where P^{μ} is the canonical momentum per particle. The final term of (4.2.21) to examine is $\underset{\sim}{t}^{\mu\nu} + \underset{\sim}{Y}^{\tau\nu\mu}{}_{|\tau}$. Using (4.1.10), (4.3.9, 10)

$$\underset{\sim}{Y}^{\tau\nu}{}_{\mu|\tau} + \underset{\sim}{t}_{\mu}{}^{\nu} = \delta^{\nu}_{\mu} \underset{\sim}{L} - \sum_{n=0}^{\infty} \Phi_{\underset{\sim}{A}|(\mu\underset{\sim}{\alpha}(n))} \underset{\sim *}{L}^{A\nu\underset{\sim}{\alpha}(n)} \tag{4.3.21}$$

$$= (\underset{\sim}{L}_1 + \underset{\sim}{L}_2)\delta^{\nu}_{\mu} - e^{(a)}_{\alpha|\mu} \frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\alpha|\nu}}$$

$$- \sum_{n=0}^{\infty} \psi_{A|(\mu\underset{\sim}{\alpha}(n))} \underset{\sim *(\tilde{\psi})}{L}^{A\nu\underset{\sim}{\alpha}(n)} - \sum_{n=0}^{\infty} R_{A|(\mu\underset{\sim}{\alpha}(n))} \underset{\sim *}{Q}^{A\nu\underset{\sim}{\alpha}(n)} \tag{4.3.22}$$

with
$$- e^{(a)}_{\alpha|\mu} \frac{\partial \underset{\sim}{L}}{\partial e^{(a)}_{\alpha|\nu}} = - e^{(a)}_{\alpha|\mu} \frac{\partial \underset{\sim}{L}_1}{\partial \dot{e}^{(a)}_{\alpha}} \underset{\sim}{N}^{\nu} \quad . \tag{4.3.23}$$

Define a canonical energy tensor for the fields ψ_A by ([14])

$$\sqrt{-g}\; t_{\mu(\psi)}{}^{\nu} \equiv \underset{\sim}{L}_2 \delta^{\nu}_{\mu} - \sum_{n=0}^{\infty} \psi_{A|(\mu\underset{\sim}{\alpha}(n))} \underset{\sim *(\tilde{\psi})}{L}^{A\nu\underset{\sim}{\alpha}(n)} \quad . \tag{4.3.24}$$

(4.3.19, 20, 22, 23 and 24) then yield

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\rho}} \underset{\sim}{N}^{\nu} g^{\rho\mu} - \frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{N}^{\alpha}} \underset{\sim}{N}^{\alpha} g^{\mu\nu} + \underset{\sim}{t}^{\mu\nu} + Y^{\tau\nu\mu}{}_{|\tau}$$

$$= P^{\mu} \underset{\sim}{N}^{\nu} + \sqrt{-g}\, t^{\mu\nu}_{(\psi)} - \sum_{n=0}^{\infty} \underset{\sim *}{Q}^{A\nu\underset{\sim}{\alpha}(n)} R_{A|(\mu\underset{\sim}{\alpha}(n))}$$

and, finally, substituting the above expression into (4.2.21) gives the final form of the gravitational field equations:

$$\boxed{\begin{aligned} -(8\pi)^{-1}G^{\mu\nu} = T^{\mu\nu} \equiv {} & n\, P^{\mu}u^{\nu} + t^{\mu\nu}_{(\psi)} + \tfrac{1}{2}(-g)^{-1/2}(\underset{\sim *}{S}^{\mu\tau\nu} + \underset{\sim *}{S}^{\nu\tau\mu} + \underset{\sim *}{S}^{\nu\mu\tau})_{|\tau} \\ & + (-g)^{-1/2}\Big(- \sum_{n=0}^{\infty} \underset{\sim *}{Q}^{A\nu\underset{\sim}{\alpha}(n)} R_{A|(\mu\underset{\sim}{\alpha}(n))} \\ & + 4\, \underset{\sim *}{Q}^{\tau(\nu\mu)\gamma}{}_{|\gamma\tau} + \underset{\sim *}{Q}^{A} (I_{A}{}^{B})^{\mu\nu} R_{B}\Big) \quad . \qquad (4.3.25) \end{aligned}}$$

In summary, the above equations generalise the equations (3.3.15) of Chapter Three, the total energy tensor $T^{\mu\nu}$ is found to contain:

(i) A material energy momentum tensor $T^{\mu\nu}_{(mat)} = n\, P^{\mu}\, u^{\nu}$ where P^{μ} is the canonical momentum per particle given by (2.3.5),

(ii) A canonical energy tensor for the fields ψ_A given by (4.3.24),

(iii) A contribution from spin, the spin flux $\underset{\sim *}{S}^{\mu\tau\nu}$ being the sum of a matter part $\underset{\sim}{S}^{\mu\tau\nu}_{(mat)} = \sqrt{-g}\, n\, S^{\mu\tau}\, u^{\nu}$ and a field part $\underset{\sim}{S}^{\mu\tau\nu}_{(\psi R)}$ that includes contributions from curvature.

(iv) Contributions from gravitational multipole moments.

CHAPTER FIVE

DIRAC EQUATION

§5.1 Discussion.

The possibility of obtaining classical relativistic equations for spinning particles starting from the Dirac equation has been explored in papers by several authors. The main attempts have been directed at deriving the Bargmann, Michel and Telegdi equations [7], by making suitable definitions for spin, four momentum, etc., either directly in terms of spinor entities, or as expectation values. More recently, Suttorp and de Groot [9] derived from the Dirac equation <u>operator</u> equations having the same form as their classical equations for a composite particle.

The two main approaches have been:

(i) to derive operator equations (see [12], and also [9]),

(ii) to apply the W.K.B. approximation to the Dirac equation (or the squared Dirac equation) [13].

In (ii) the lowest order equations have similar form to the Bargmann, Michel and Telegdi equations.

None of the references really seems to answer the question of how one should define relativistic objects, such as tensors describing spin and momentum, in terms of quantum mechanical entities. This same difficulty arises in §5.3 where <u>exact</u> equations are quite easily obtained from the general relativistic Dirac equation, equations having the same formal

appearance as equations (2.4.1) and (2.4.2). Section §5.2 will briefly describe the necessary notation and four-spinor calculus.

§5.2 Four-Spinor Calculus.

Let γ_a $(a = 1,\ldots,4)$ denote a fixed set of Dirac matrices satisfying

$$\gamma_{(a}\gamma_{b)} = \eta_{ab} = \mathrm{diag}\,(-1,-1,-1,1) \quad . \tag{5.2.1}$$

If γ^+ denotes the hermitian conjugate of γ, then

$$\gamma_a^+ = -\gamma_a \ (a=1,\ldots,3) \quad , \quad \gamma_4^+ = \gamma_4 \Rightarrow \gamma_4\gamma_a^+\gamma_4 = \gamma_a \quad (a=1,\ldots,4) \tag{5.2.2}$$

Define

$$\sigma_{ab} \equiv \gamma_{[a}\gamma_{b]} \tag{5.2.3}$$

then

$$\gamma_a\,\sigma_{bc} - \sigma_{bc}\gamma_a = 4\,\eta_{a[b}\gamma_{c]} \quad . \tag{5.2.4}$$

If $\omega^{ab} = -\,\omega^{ba}$ is an arbitrary set of scalars (1×1 matrices) then (5.2.4) implies

$$\omega_a{}^c\,\gamma_c = \frac{1}{4}\,(\gamma_a\sigma_{bc} - \sigma_{bc}\gamma_a)\,\omega^{bc} \quad . \tag{5.2.5}$$

For an o.n field of tetrads $e^{(a)}_\mu(x)$:

$$\eta_{ab}\, e_\mu^{(a)}\, e_\nu^{(b)} = g_{\mu\nu}(x) \tag{5.2.6}$$

the Ricci rotation coefficients are

$$\gamma^a{}_{bc} = -e^{(a)}_{\alpha|\beta}\, e^\alpha_{(b)}\, e^\beta_{(c)} \quad . \tag{5.2.7}$$

Note

$$\gamma_{abc} = \gamma_{[ab]c} \tag{5.2.8}$$

$$e^{(a)}_{\alpha|\beta} = -\gamma^a{}_{bc}\, e^{(b)}_\alpha\, e^{(c)}_\beta \quad . \tag{5.2.9}$$

<u>Field of</u> $\gamma_\mu(x)$. For <u>fixed</u> numerical matrices γ_a define

$$\gamma_\mu(x) \equiv \gamma_a\, e^{(a)}_\mu(x) \quad . \tag{5.2.10}$$

(5.2.1 and 6) imply

$$\gamma_{(\mu}\gamma_{\nu)} = g_{\mu\nu} \quad . \tag{5.2.11}$$

(5.2.9 and 10) imply

$$\begin{aligned} \gamma_{\mu|\nu} &\equiv \gamma_a\, e^{(a)}_{\mu|\nu} \\ &= -\gamma_a\, \gamma^a{}_{bc}\, e^{(b)}_\mu\, e^{(c)}_\nu \quad . \end{aligned}$$

Write this as

$$e_{(b)}{}^\mu\, \gamma_{\mu|\nu} dx^\nu = -\gamma_a\, \omega^a{}_b \ , \ \omega^a{}_b \equiv \gamma^a{}_{bc}\, e^{(c)}_\nu\, dx^\nu \quad .$$

then $$e_{(b)}{}^{\mu}\, \gamma_{\mu|\nu} dx^{\nu} = \frac{1}{4}\,(\gamma_b \sigma_{cd} - \sigma_{cd}\gamma_b)\,\omega^{cd} \qquad \text{by (5.2.5)}$$

i.e.,

$$\gamma_{\mu|\nu} = -\,(\gamma_{\mu}\Gamma_{\nu} - \Gamma_{\nu}\gamma_{\mu}) \tag{5.2.12}$$

$$\Gamma_{\nu} \equiv -\frac{1}{4}\,\sigma_{cd}\,\gamma^{cd}{}_{f}\, e_{\nu}^{(f)} \quad . \tag{5.2.13}$$

Define

$$\nabla_{\nu}\gamma_{\mu} \equiv \gamma_{\mu|\nu} + [\gamma_{\mu},\Gamma_{\nu}] \tag{5.2.14}$$

then

$$\nabla_{\nu}\gamma_{\mu} = 0 \quad . \tag{5.2.15}$$

The covariant derivative of a spinor ψ is defined so that $\gamma^{\mu}\nabla_{\mu}\psi + k\psi = 0$ is invariant under co-ordinate transformations, and also invariant under arbitrary tetrad rotations

$$e_{\mu}^{(a)\,'} = \Lambda^{a}{}_{b}\;(x)\; e_{\mu}^{(b)} \tag{5.2.16}$$

which can vary from point to point. From (5.2.16) and (5.2.10)

$$\gamma'_{\mu}\;(x) = \Lambda_{\mu}{}^{\nu}(x)\;\gamma_{\nu}(x) \tag{5.2.17}$$

where $\Lambda_{\mu}{}^{\nu} = e^{\nu}_{(a)}\;\Lambda^{a}{}_{b}\; e_{\mu}^{(b)}$. Since $\gamma'_{(\mu}\gamma'_{\nu)} = g_{\mu\nu}$ by Pauli's theorem, there exists a 4×4 matrix $S(\Lambda)$ such that

$$\gamma'_\mu = S\ \gamma_\mu\ S^{-1} \quad . \tag{5.2.18}$$

Postulate that under tetrad rotation (5.2.16) ψ transforms as

$$\psi'(x) = S(x)\ \psi(x) \quad . \tag{5.2.19}$$

(This ensures that for constant-tetrad rotations (e.g. special relativity) $\gamma^\mu\ \partial_\mu\psi + k\psi = 0$ is invariant.) Then

$$\begin{aligned}\gamma^{\mu'}\partial_\mu\psi' &= S\ \gamma^\mu\ S^{-1}\ \partial_\mu(S\psi)\\ &= S(\gamma^\mu\partial_\mu\psi + \gamma^\mu(S^{-1}\partial_\mu S)\psi)\\ &= S\ \gamma^\mu(\partial_\mu\psi + (S^{-1}\partial_\mu S)\psi) \quad .\end{aligned}$$

Now, the transformation law for Γ_μ under (5.2.16) is

$$\Gamma'_\mu = S\ \Gamma_\mu\ S^{-1} + (\partial_\mu\ S)S^{-1} \quad . \tag{5.2.20}$$

This can be computed from (5.2.16, 13 and 7), or, more easily, from $\gamma_{\mu|\nu} = -[\gamma_\mu,\Gamma_\nu]$ and (5.2.18). Hence, defining

$$\nabla_\mu\psi \equiv \partial_\mu\psi - \Gamma_\mu\ \psi \quad (\psi{=}4{\times}1 \text{ spinor}) \tag{5.2.21}$$

we achieve invariance of $\gamma^\mu\nabla_\mu\psi + k\psi = 0$. For the Pauli conjugate spinor $\overline{\psi} \equiv \psi^+\gamma_4$, define ∇_μ acting on Pauli conjugate $\overline{\psi}$ by $\nabla_\mu(\overline{\psi}) \equiv (\overline{\nabla_\mu\psi})$. Then (5.2.21) gives

$$\nabla_\mu\overline{\psi} = \partial_\mu\overline{\psi} - \psi^+\ \Gamma^+_\mu\ \gamma_4$$

$$= \partial_\mu \overline{\psi} - \overline{\psi}\,(\gamma_4 \Gamma_\mu^+ \gamma_4)$$

$$= \partial_\mu \overline{\psi} + \overline{\psi}\,\Gamma_\mu$$

by (5.2.2, 3 and 13). Hence, for 1×4 spinors

$$\nabla_\mu \phi \equiv \partial_\mu \phi + \phi\,\Gamma_\mu \quad (\phi = 1\times 4 \text{ spinor}) \quad . \tag{5.2.22}$$

For tensor-spinors (4×4) such as γ^μ, $\sigma^{\mu\nu}$ etc.,

$$\nabla_\mu T^{\kappa\lambda} = T^{\kappa\lambda}{}_{|\mu} + [T^{\kappa\lambda}, \Gamma_\mu] \quad . \tag{5.2.23}$$

All these covariant derivatives transform as <u>tensors</u> under <u>co-ordinate</u> transformations, and are <u>covariant</u> under <u>tetrad rotations</u> in the sense $(\nabla_\mu \psi)' = S(\nabla_\mu \psi)$, $(\nabla_\mu \phi)' = (\nabla_\mu \phi)\,S^{-1}$ etc. The rule (5.2.23) is dictated by the requirements

(i) that the product rule hold for ∇_μ ;

(ii) any spin tensor $\underset{(4\times4)}{T^{\kappa\lambda}} = \underset{(4\times1)}{\psi}\;\underset{(1\times4)}{\phi}\;\underset{(1\times1)}{T^{\kappa\lambda}}$ or sum of such terms;

(iii) for plain tensor $T^{\kappa\lambda}$, $\nabla_\mu T^{\kappa\lambda} = T^{\kappa\lambda}{}_{|\mu}$.

Identity (5.2.15) is unaltered by replacing Γ_μ , ∇_μ by $\tilde{\Gamma}_\mu$, $\tilde{\nabla}_\mu$, where $\tilde{\Gamma}_\mu = \Gamma_\mu + i\,\underset{(1\times1)}{A_\mu}\; I$. (Factor i is included to preserve identity $\gamma_4\,\Gamma_\mu^+ \gamma_4 = -\Gamma_\mu$ (A_μ real vector).) The Ricci commutation relations take the form

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\psi = \Phi_{\mu\nu}\,\psi \tag{5.2.24}$$

where $$\Phi_{\mu\nu} = 2\,\partial_{[\mu}\Gamma_{\nu]} - [\Gamma_\mu,\Gamma_\nu] \tag{5.2.25}$$

$$= -\frac{1}{4} R_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta} + 2i\,\partial_{[\mu}A_{\nu]} \quad . \tag{5.2.26}$$

§5.3 Dirac Equation in General Relativity.

The general relativistic Dirac equation, including an anomalous magnetic moment term is

$$i\,\gamma^\mu\nabla_\mu\psi + (m/h)\psi + i\,g\,F_{\mu\nu}\gamma^\mu\gamma^\nu\,\psi = 0 \quad . \tag{5.3.1}$$

The hermitian conjugate equation is

$$-i(\nabla_\mu\psi)^+\gamma^{\mu+} + (m/h)\psi^+ - i\,g\,F_{\mu\nu}\psi^+\gamma^{\nu+}\gamma^{\mu+} = 0 \quad .$$

Multiplication on the right by $-\gamma_4$ and insertion of $\gamma_4\gamma_4 = 1$ in appropriate places gives

$$+\,i(\nabla_\mu\psi)^+\gamma_4\gamma_4\gamma^{\mu+}\gamma_4 - (m/h)\overline{\psi} + i\,g\,F_{\mu\nu}\overline{\psi}\gamma_4\gamma^{\nu+}\gamma_4\gamma_4\gamma^{\mu+}\gamma_4 = 0$$

and use of (5.2.1) and (5.2.2) simplifies this to the Pauli conjugate of (5.3.1)

$$i(\nabla_\mu\overline{\psi})\gamma^\mu - (m/h)\overline{\psi} + i\,g\,F_{\mu\nu}\,\overline{\psi}\,\gamma^\nu\gamma^\mu = 0 \quad . \tag{5.3.2}$$

Define

$$T^{\mu\nu} \equiv \overline{\psi}\,\nabla^\mu\gamma^\nu\,\psi \quad . \tag{5.3.3}$$

$T^{\mu\nu}$ is a tensor (1×1 matrix) so $\nabla_\mu T^{\mu\nu} = T^{\mu\nu}{}_{|\mu}$. Using (5.3.1), (5.3.2) and (5.2.24, 25, 26) gives

$$\nabla_\nu T^{\mu\nu} = \nabla_\nu(\overline{\psi}\nabla^\mu\gamma^\nu\psi)$$

$$= (\nabla_\nu\overline{\psi})\nabla^\mu\gamma^\nu\psi + \overline{\psi}(\nabla_\nu\nabla^\mu\gamma^\nu\psi)$$

$$= (\nabla_\nu\overline{\psi})\gamma^\nu(\nabla^\mu\psi) + \overline{\psi}(\nabla^\mu\nabla_\nu + \Phi^\mu{}_\nu)\gamma^\nu\,\psi$$

$$= ((m/ih)\overline{\psi} - g\,F_{\alpha\beta}\,\overline{\psi}\gamma^\beta\gamma^\alpha)(\nabla^\mu\psi)$$

$$+ \overline{\psi}\nabla^\mu(-(m/ih)\psi - g\,F_{\alpha\beta}\gamma^\alpha\gamma^\beta\psi)$$

$$+ \overline{\psi}(-\frac{1}{4}R^\mu{}_{\nu\alpha\beta}\sigma^{\alpha\beta} + i\,F^\mu{}_\nu)\gamma^\nu\psi\ .$$

Cancellation of terms gives

$$\nabla_\nu T^{\mu\nu} = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}(\overline{\psi}\sigma^{\alpha\beta}\gamma^\nu\psi) + F^\mu{}_\nu(\overline{\psi}\,i\,\gamma^\nu\psi) - g\,F_{\alpha\beta}{}^{|\mu}(\overline{\psi}\sigma^{\alpha\beta}\psi) \quad . \ (5.3.4)$$

Equation (5.3.4) has the same form as (2.4.2). Defining

$$eu^\nu \equiv i\,\overline{\psi}\,\gamma^\nu\psi\ ;\quad M^{\alpha\beta} \equiv -2g\,\overline{\psi}\,\sigma^{\alpha\beta}\psi$$

and

$$S^{\alpha\beta\nu} \equiv \frac{1}{2}\overline{\psi}\,\sigma^{\alpha\beta}\gamma^\nu\psi$$

gives the R.H.S. of (5.3.4) identical to the R.H.S. of (2.4.2). For spin equations, we take the divergence of $S^{\alpha\beta\nu}$.

$$\nabla_\lambda\, S^{\alpha\beta\lambda} = \frac{1}{2}\nabla_\lambda(\overline{\psi}\sigma^{\alpha\beta}\gamma^\lambda\psi)$$

$$= \frac{1}{2}\,(\nabla_\lambda\overline{\psi})\sigma^{\alpha\beta}\gamma^\lambda\psi + \frac{1}{2}\,\overline{\psi}\sigma^{\alpha\beta}\gamma^\lambda\nabla_\lambda\psi$$

$$= \frac{1}{2}\,(\nabla_\lambda\overline{\psi})\,(\gamma^\lambda\sigma^{\alpha\beta} - 4g^{\lambda[\alpha}\gamma^{\beta]})\psi + \frac{1}{2}\,\overline{\psi}\,\sigma^{\alpha\beta}\gamma^\lambda\nabla_\lambda\psi$$

$$= \frac{1}{2}\,(\frac{m}{ih}\,\overline{\psi} - g\,F_{\mu\nu}\,\overline{\psi}\gamma^\nu\gamma^\mu)\sigma^{\alpha\beta}\psi$$

$$- 2(\nabla^{[\alpha}\overline{\psi})\gamma^{\beta]}\psi + \frac{1}{2}\,\overline{\psi}\,\sigma^{\alpha\beta}(-\frac{m}{ih}\,\psi - g\,F_{\mu\nu}\,\gamma^\mu\gamma^\nu\psi) \quad .$$

Cancelling the $\overline{\psi}\,\sigma^{\alpha\beta}\psi$ terms, and noticing that

$$F_{\mu\nu}\,\gamma^\nu\gamma^\mu\sigma^{\alpha\beta} + F_{\mu\nu}\,\sigma^{\alpha\beta}\gamma^\mu\gamma^\nu = 8\,F_\mu{}^{[\alpha}\sigma^{\beta]\mu} \quad ,$$

gives
$$\nabla_\lambda\, S^{\alpha\beta\lambda} = -2(\nabla^{[\alpha}\overline{\psi})\gamma^{\beta]}\psi - 4\,g\,F_\mu{}^{[\alpha}\overline{\psi}\sigma^{\beta]\mu}\psi \; .$$

Defining $P^{\alpha\beta}$ by $P^{\alpha\beta} \equiv -(\nabla^\alpha\overline{\psi})\gamma^\beta\psi$ gives the above equations as

$$\frac{1}{2}\nabla_\lambda\, S^{\alpha\beta\lambda} = P^{[\alpha\beta]} - F^{[\alpha}{}_\mu M^{\beta]\mu} \tag{5.3.5}$$

which may be compared with spin equation (2.4.1). The question of interpretation of (5.3.4) and (5.3.5) will not be pursued further, except for remarking that a quantum theorist might demand definitions of u^λ, $M^{\mu\nu}$ etc., as expectation values rather than 1×1 spinors. $T^{\mu\nu}$ is a (canonical) energy tensor, and one may notice that, in geodesic co-ordinates, integration of (5.3.4) over 3-space (i.e. taking expectation values of the operators $\sigma^{\alpha\beta}\gamma^\nu$, $i\,\gamma^\nu$, $\sigma^{\alpha\beta}$) gives for the LHS of (5.3.4)

$$\int T^{\mu\nu}{}_{|\nu}\, d^3x = \int T^{\mu\nu}{}_{,\nu}\, d^3x$$

$$= \int T^{\mu i}{}_{,i}\, d^3x + \int T^{\mu 4}{}_{,4}\, d^3x \qquad (i = 1,2,3)$$

$$= \int (\partial T^{\mu 4}/\partial t)\, d^3x \qquad \text{(by Gauss's Theorem)}$$

$$= \frac{\partial P^{\mu}}{\partial t} \qquad \left(P^{\mu} \equiv \int T^{\mu 4}\, d^3x\right) .$$

This is valid in special relativity and may be assumed in general relativity for small d^3x .

APPENDIX ONE

CO-ORDINATE TRANSFORMATIONS

Consider a general co-ordinate transformation $\bar{x}^\rho = \bar{x}^\rho(x^\lambda)$. Define $X^\rho{}_\sigma \equiv \partial\bar{x}^\rho/\partial x^\sigma$. Since

$$\frac{\partial\bar{x}^\sigma}{\partial x^s}\frac{\partial x^s}{\partial\bar{x}^\rho} = \delta^\sigma_\rho \quad ,$$

the inverse matrix is $X^{-1\sigma}{}_\rho = \partial x^\sigma/\partial\bar{x}^\rho$.

A relative tensor $\Phi_A(x)$ will transform according to

$$\bar{\Phi}_A(\bar{x}) = \Lambda_A{}^B\,(X^\rho{}_\sigma)\,\Phi_B(x) \quad . \tag{A.1.1}$$

Here A denotes the collection of indices, contravariant and covariant, of Φ_A , and repeated capital indices indicates summation over each index of A . All indices will run from one to four, unless otherwise indicated. For a scalar

$$\Phi_A = \phi \ , \quad \text{we have} \quad \Lambda_A{}^B = 1 \quad ,$$

for a contravariant vector

$$\Phi_A = \phi^\alpha \quad \text{we have} \quad \Lambda_A{}^B = X^\alpha{}_\beta \quad ,$$

for a covariant vector

$$\Phi_A = \phi_a \quad \text{we have} \quad \Lambda_A{}^B = X^{-1b}{}_a$$

and generally for a relative tensor of weight w

$$\Phi_A = \Phi^{\alpha_1 \dots \alpha_m}_{a_1 \dots a_n}(x) \quad ,$$

$\Lambda_A{}^B$ is given by

$$\Lambda_A{}^B = X^{\alpha_1}_{\beta_1} \dots X^{\alpha_m}_{\beta_m} X^{-1 b_1}{}_{a_1} \dots X^{-1 b_n}{}_{a_n} |X^\rho{}_\sigma|^{-w} \quad . \qquad \text{(A.1.2)}$$

Define infinitesimal generators $(I_A{}^B)_\rho{}^\sigma$ by

$$(I_A{}^B)_\rho{}^\sigma \equiv \frac{\partial \Lambda_A{}^B}{\partial X^\rho{}_\sigma} \quad \text{(evaluated at } X^\rho{}_\sigma = \delta^\rho{}_\sigma \text{)} \quad . \qquad \text{(A.1.3)}$$

The $(I_A{}^B)_\rho{}^\sigma$'s may be constructed explicitly from recursion formulae:

For a relative invariant ϕ of weight w ,

$$\text{i.e.} \quad \bar{\phi}(\bar{x}) = \left|\frac{\partial x^\sigma}{\partial \bar{x}^\rho}\right|^w \phi(x) \text{ , we have } \bar{\phi} = |X|^{-w}\phi = \Lambda\phi \quad ,$$

$$\therefore \quad (I)_\rho{}^\sigma \equiv \frac{\partial \Lambda}{\partial X^\rho{}_\sigma} (X^\rho{}_\sigma = \delta^\rho{}_\sigma) = -w|X|^{-w-1} \frac{\partial |X|}{\partial X^\rho{}_\sigma} (X^\rho{}_\sigma = \delta^\rho{}_\sigma)$$

$$= -w \, \delta^\sigma_\rho \quad . \qquad \text{(A.1.4)}$$

Differentiating $X^\alpha{}_\beta X^{-1\beta}{}_\gamma = \delta^\alpha_\gamma$ w.r. to $X^\rho{}_\sigma$ gives

$$\delta^\alpha_\rho \delta^\sigma_\beta X^{-1\beta}{}_\gamma + X^\alpha{}_\beta \frac{\partial}{\partial X^\rho{}_\sigma} (X^{-1\beta}{}_\gamma) = 0 \quad .$$

Multiplying by $X^{-1\lambda}{}_\alpha$ gives $\dfrac{\partial}{\partial X^\rho{}_\sigma}(X^{-1\lambda}{}_\gamma) = -X^{-1\lambda}{}_\rho X^{-1\sigma}{}_\gamma$. Hence

$$\frac{\partial}{\partial X^\rho{}_\sigma} (X^{-1\lambda}{}_\gamma) \; (X^\rho{}_\sigma = \delta^\rho_\sigma) = - \delta^\lambda_\rho \delta^\sigma_\gamma \quad . \qquad \text{(A.1.5)}$$

For a relative tensor Φ_A transforming as $\bar{\Phi}_A(\bar{x}) = \Lambda_A{}^B\, \Phi_B(x)$, a relative tensor $\Phi_{A\alpha}$ with an additional covariant index α, will transform as

$$\bar{\Phi}_{A\alpha}(\bar{x}) = \Lambda_{A\alpha}{}^{B\beta}(X^\rho{}_\sigma)\, \Phi_{B\beta}(x) = \Lambda_A{}^B\, X^{-1\beta}{}_\alpha\, \Phi_{B\beta}(x)$$

$$\therefore \quad (I_{A\alpha}{}^{B\beta})_\rho{}^\sigma = \frac{\partial}{\partial X^\rho{}_\sigma}(\Lambda_A{}^B\, X^{-1\beta}{}_\alpha) \quad (X^\rho{}_\sigma = \delta^\rho{}_\sigma)$$

$$\text{i.e.,} \quad (I_{A\alpha}{}^{B\beta})_\rho{}^\sigma = (I_A{}^B)_\rho{}^\sigma\, \delta^\beta_\alpha - \delta^B_A\, \delta^\beta_\rho\, \delta^\sigma_\alpha \quad . \qquad \text{(A.1.6(a))}$$

Similarly, for an extra contra-variant index,

$$\bar{\Phi}^\alpha_A(\bar{x}) = \Lambda_A{}^B\, X^\alpha{}_\beta\, \Phi^\beta_B(x)$$

$$\therefore \quad (I^{\alpha B}_{A\beta})_\rho{}^\sigma = \frac{\partial}{\partial X^\rho{}_\sigma}(\Lambda_A{}^B\, X^\alpha{}_\beta) \quad (X^\rho{}_\sigma = \delta^\rho{}_\sigma)$$

$$\text{i.e.} \quad (I^{\alpha B}_{A\beta})_\rho{}^\sigma = (I_A{}^B)_\rho{}^\sigma\, \delta^\alpha_\beta + \delta^B_A\, \delta^\alpha_\rho\, \delta^\sigma_\beta \quad . \qquad \text{(A.1.6(b))}$$

From the group property of transformations, $\Lambda_A{}^B(Y^\tau{}_\rho X^\rho{}_\sigma) = \Lambda_A{}^C(Y)\, \Lambda_C{}^B(X)$, by differentiation we have

$$(I_A{}^C)_\lambda{}^\mu (I_C{}^B)_\alpha{}^\beta - (I_A{}^C)_\alpha{}^\beta (I_C{}^B)_\lambda{}^\mu = (I_A{}^B)_\lambda{}^\beta\, \delta^\mu_\alpha - (I_A{}^B)_\alpha{}^\mu\, \delta^\beta_\lambda \ . \qquad \text{(A.1.7)}$$

The main usefulness of $(I_A{}^B)_\rho{}^\sigma$ is that covariant differentiations, and variations on co-ordinate transformation, of a relative tensor Φ_A can be succinctly written down.

For an infinitesimal transformation $\bar{x}^\rho = x^\rho + \xi^\rho(x)$, $X^\rho{}_\sigma = \partial\bar{x}^\rho/\partial x^\sigma = \delta^\rho{}_\sigma + \xi^\rho{}_{,\sigma}$. Therefore a relative tensor Φ_A will transform according to

$$\bar{\Phi}_A(\bar{x}) = \Lambda_A{}^B(\delta^\rho{}_\sigma + \xi^\rho{}_{,\sigma})\ \Phi_B(x)$$

$$= \Lambda_A{}^B(\delta^\rho{}_\sigma)\Phi_B(x) + \frac{\partial\Lambda_A{}^B}{\partial X^\rho{}_\sigma}\ (X^\rho{}_\sigma = \delta^\rho{}_\sigma)\ (\xi^\rho{}_{,\sigma})\ \Phi_B(x)$$

$$\text{i.e.}\quad \bar{\Phi}_A(\bar{x}) = \Phi_A(x) + (I_A{}^B)_\rho{}^\sigma\ (\xi^\rho{}_{,\sigma})\ \Phi_B(x)\quad . \tag{A.1.8}$$

$(I_A{}^B)_\rho{}^\sigma$ can be written down explicitly: For a relative tensor $\Phi_A = \Phi^{\alpha_1 \ldots \alpha_m}{}_{a_1 \ldots a_n}$,

$$\Lambda_A{}^B = X^{\alpha_1}_{\beta_1} \ldots X^{\alpha_m}_{\beta_m}\, X^{-1}{}^{b_1}{}_{a_1} \ldots X^{-1}{}^{b_n}{}_{a_n}\, |X|^{-w}$$

$$\therefore\ \frac{\partial\Lambda_A{}^B}{\partial X^\rho{}_\sigma} = (\sum_{i=1}^m X^{\alpha_1}_{\beta_1} \ldots X^{\alpha_{i-1}}_{\beta_{i-1}}\, (\delta^{\alpha_i}_\rho \delta^\sigma_{\beta_i})X^{\alpha_{i+1}}_{\beta_{i+1}} \ldots X^{\alpha_m}_{\beta_m}\, X^{-1}{}^{b_1}{}_{a_1} \ldots X^{-1}{}^{b_n}{}_{a_n}$$

$$+ \sum_{j=1}^n X^{\alpha_1}_{\beta_1} \ldots X^{\alpha_m}_{\beta_m}\, X^{-1}{}^{b_1}{}_{a_1} \ldots X^{-1}{}^{b_{j-1}}{}_{a_{j-1}}(-X^{-1}{}^{b_j}{}_\rho X^{-1\sigma}{}_{a_j})X^{-1}{}^{b_{j+1}}{}_{a_{j+1}} \ldots X^{-1}{}^{b_n}{}_{a_n})|X|^{-w}$$

$$+ (X^{\alpha_1}_{\beta_1} \ldots X^{\alpha_m}_{\beta_m}\, X^{-1}{}^{b_1}{}_{a_1} \ldots X^{-1}{}^{b_n}{}_{a_n})(-w)\ |X|^{-w-1}\,\frac{\partial|X|}{\partial X^\rho{}_\sigma}\quad .$$

$$\therefore\ (I_A{}^B)_\rho{}^\sigma = \frac{\partial\Lambda_A{}^B}{\partial X^\rho{}_\sigma}\ (X^\rho{}_\sigma = \delta^\rho{}_\sigma)$$

$$= \sum_{i=1}^m \delta^{\alpha_1}_{\beta_1} \ldots \delta^{\alpha_{i-1}}_{\beta_{i-1}}\, (\delta^{\alpha_i}_\rho \delta^\sigma_{\beta_i})\ \delta^{\alpha_{i+1}}_{\beta_{i+1}} \ldots \delta^{\alpha_m}_{\beta_m}\, \delta^{b_1}_{a_1} \ldots \delta^{b_n}_{a_n}$$

$$- \sum_{j=1}^{n} \delta^{\alpha_1}_{\beta_1} \cdots \delta^{\alpha_m}_{\beta_m} \delta^{b_1}_{a_1} \cdots \delta^{b_{j-1}}_{a_{j-1}} (\delta^{b_j}_{\rho} \delta^{\sigma}_{a_j}) \delta^{b_{j+1}}_{a_{j+1}} \cdots \delta^{b_n}_{a_n} \qquad \text{(A.1.9)}$$

$$-w \, \delta^{B}_{A} \, \delta^{\sigma}_{\rho} \quad .$$

We can write $\Phi_{A|\tau}$ conveniently in terms of $(I_A{}^B)_\rho{}^\sigma$:

$$\Phi_{A|\tau} = \partial_\tau \Phi^{\alpha_1 \ldots \alpha_m}_{a_1 \ldots a_n} + \sum_{i=1}^{m} \Gamma^{\alpha_i}_{\beta_i \tau} \Phi^{\alpha_1 \ldots \alpha_{i-1} \beta_i \alpha_{i+1} \ldots \alpha_m}_{a_1 \ldots \qquad \ldots a_n}$$

$$- \sum_{j=1}^{n} \Gamma^{b_j}_{a_j \tau} \Phi^{\alpha_1 \ldots \qquad \ldots \alpha_m}_{a_1 \ldots a_{j-1} b_j a_{j+1} \ldots a_n} - w \, \Gamma^{\lambda}_{\lambda\tau} \Phi_A$$

$$= \partial_\tau \Phi_A + \Gamma^{\rho}{}_{\sigma\tau} \Big(\sum_{i=1}^{m} \delta^{\alpha_1}_{\beta_1} \cdots \delta^{\alpha_{i-1}}_{\beta_{i-1}} (\delta^{\alpha_i}_{\rho} \delta^{\sigma}_{\beta_i}) \delta^{\alpha_{i+1}}_{\beta_{i+1}} \cdots \delta^{\alpha_m}_{\beta_m} \delta^{b_1}_{a_1} \cdots \delta^{b_n}_{a_n}$$

$$- \sum_{j=1}^{n} \delta^{\alpha_1}_{\beta_1} \cdots \delta^{\alpha_m}_{\beta_m} \delta^{b_1}_{a_1} \cdots \delta^{b_{j-1}}_{a_{j-1}} (\delta^{b_j}_{\rho} \delta^{\sigma}_{a_j}) \delta^{b_{j+1}}_{a_{j+1}} \cdots \delta^{b_n}_{a_n} \Big) \Phi^{\beta_1 \ldots \beta_m}_{b_1 \ldots b_n}$$

$$+ \Gamma^{\rho}{}_{\sigma\tau} \, (-w \, \delta^{\sigma}_{\rho} \delta^{B}_{A}) \, \Phi_B \quad .$$

Comparison with (A.1.9) then gives

$$\Phi_{A|\tau} = \partial_\tau \Phi_A + \Gamma^{\rho}{}_{\sigma\tau} \, (I_A{}^B)_\rho{}^\sigma \, \Phi_B \quad . \qquad \text{(A.1.10)}$$

In geodesic co-ordinates, with $\Gamma^{\alpha}_{\beta\gamma} = 0$,

$$\Phi_{A|\mu\nu} = (\Phi_{A|\mu})_{,\nu} = (\partial_\mu \Phi_A - \Gamma^{\rho}{}_{\sigma\mu} (I_A{}^B)_\rho{}^\sigma \, \Phi_B)_{,\nu}$$

$$= \partial_\nu \partial_\mu \Phi_A + \Gamma^{\rho}{}_{\sigma\mu,\nu} (I_A{}^B)_\rho{}^\sigma \, \Phi_B$$

$$\Phi_{A|\mu\nu} - \Phi_{A|\nu\mu} = (\Gamma^{\rho}{}_{\sigma\mu,\nu} - \Gamma^{\rho}{}_{\sigma\nu,\mu})(I_A{}^B)_{\rho}{}^{\sigma}\,\Phi_B$$

$$= -R^{\rho}{}_{\sigma\mu\nu}\,(I_A{}^B)_{\rho}{}^{\sigma}\,\Phi_B \quad .$$

So, in any co-ordinate system,

$$\Phi_{A|\mu\nu} - \Phi_{A|\nu\mu} = -R^{\rho}{}_{\sigma\mu\nu}\,(I_A{}^B)_{\rho}{}^{\sigma}\,\Phi_B \quad . \tag{A.1.11}$$

APPENDIX TWO

CO-ORDINATE INVARIANCE

We write down the identities resulting from the relative invariance, under arbitrary co-ordinate transformations, of scalar densities dependent upon tensor arguements.

Basic Identity. Suppose $\underset{\sim}{L}(\underset{\sim}{\Phi_A})$ is a relative invariant of weight w,

$$\underset{\sim}{L}(\underset{\sim}{\overline{\Phi}_A}(\overline{x})) = \left|\frac{\partial x^\sigma}{\partial \overline{x}^\rho}\right|^w \underset{\sim}{L}(\underset{\sim}{\Phi_A}(x)) \quad . \tag{A.2.1}$$

Making an arbitrary infinitesimal transformation $\overline{x}^\rho = x^\rho + \xi^\rho(x)$, to first order we have

$$\partial x^\sigma/\partial \overline{x}^\rho = \delta^\sigma_\rho - \xi^\sigma{}_{,\rho} \quad ; \quad \left|\frac{\partial x^\sigma}{\partial \overline{x}^\rho}\right| = 1 - \xi^\sigma{}_{,\sigma} \quad ; \quad \left|\frac{\partial x^\sigma}{\partial \overline{x}^\rho}\right|^w = 1 - w\xi^\sigma{}_{,\sigma} \quad ;$$

giving (A.2.1) as

$$\underset{\sim}{L}(\underset{\sim}{\overline{\Phi}_A}(\overline{x})) - \underset{\sim}{L}(\underset{\sim}{\Phi_A}(x)) = -w\xi^\sigma{}_{,\sigma} \underset{\sim}{L} \quad . \tag{A.2.2}$$

Noticing (A.1.8), $\underset{\sim}{\overline{\Phi}_A}(\overline{x}) - \underset{\sim}{\Phi_A}(x) = (\underset{\sim}{I_A{}^B})_\rho{}^\sigma \underset{\sim}{\Phi_B}(\xi^\rho{}_{,\sigma})$, gives

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{\Phi_A}} (\underset{\sim}{I_A{}^B})_\rho{}^\sigma \underset{\sim}{\Phi_B}(\xi^\rho{}_{,\sigma}) = -w\delta^\sigma_\rho \underset{\sim}{L}(\xi^\rho{}_{,\sigma}) \quad \text{for arbitrary} \quad \xi^\rho(x) \quad .$$

$\underset{\sim}{L}$ therefore satisfies the basic identity

$$\frac{\partial \underset{\sim}{L}}{\partial \underset{\sim}{\Phi_A}} (\underset{\sim}{I_A{}^B})_\rho{}^\sigma \underset{\sim}{\Phi_B} + w\delta^\sigma_\rho \underset{\sim}{L} = 0 \quad . \tag{A.2.3}$$

As a special case of the above, take $\underset{\sim}{\Phi_A}$ to consist of a set of tensors Φ_A and their first covariant derivatives $\Phi_{A|\alpha}$. If $\underset{\sim}{L}$ is a scalar density, $w = 1$, (A.2.3) is then

$$\frac{\partial \underset{\sim}{L}}{\partial \Phi_A} (I_A{}^B)_\rho{}^\sigma \Phi_B + \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}} (I_{A\alpha}{}^{B\beta})_\rho{}^\sigma \Phi_{B|\beta} + \delta^\sigma_\rho \underset{\sim}{L} = 0 \quad .$$

Using (A.1.6(a)),

$$\frac{\partial \underset{\sim}{L}}{\partial \Phi_A} (I_A{}^B)_\rho{}^\sigma \Phi_B + \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}} (I_A{}^B)_\rho{}^\sigma \Phi_{B|\alpha} - \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\sigma}} \Phi_{A|\rho} + \delta^\sigma_\rho \underset{\sim}{L} = 0 \quad .$$

Rearranging terms,

$$\left(\frac{\partial \underset{\sim}{L}}{\partial \Phi_A} - \nabla_\alpha \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}}\right) (I_A{}^B)_\rho{}^\sigma \Phi_B + \left(\frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}} (I_A{}^B)_\rho{}^\sigma \Phi_B\right)_{|\alpha}$$

$$- \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\sigma}} \Phi_{A|\rho} + \delta^\sigma_\rho \underset{\sim}{L} = 0$$

i.e.,

$$\frac{\delta \underset{\sim}{L}}{\delta \Phi_A} (I_A{}^B)_\rho{}^\sigma \Phi_B + \underset{\sim}{U}^{\alpha\sigma}{}_{\rho|\alpha} + \underset{\sim}{t}_\rho{}^\sigma = 0 \qquad \text{(A.2.4)}$$

where we define

$$\underset{\sim}{L}^A \equiv \frac{\partial \underset{\sim}{L}}{\partial \Phi_A} \; ; \; \underset{\sim}{L}^{A\alpha} \equiv \frac{\partial \underset{\sim}{L}}{\partial \Phi_{A|\alpha}} \; ; \; \frac{\delta \underset{\sim}{L}}{\delta \Phi_A} \equiv \underset{\sim}{L}^A - \underset{\sim}{L}^{A\alpha}{}_{|\alpha} \; ;$$

$$\underset{\sim}{U}^{\alpha\sigma}{}_\rho \equiv \underset{\sim}{L}^{A\alpha} (I_A{}^B)_\rho{}^\sigma \Phi_B \; ; \; \underset{\sim}{t}_\rho{}^\sigma \equiv \underset{\sim}{L}\, \delta^\sigma_\rho - \Phi_{A|\rho} \underset{\sim}{L}^{A\sigma} \quad .$$

APPENDIX THREE

DEFINITION OF SPIN

Consider the simple example in classical mechanics of a sphere rotating about its centre, with kinetic energy of rotation $\frac{1}{2} I w^2$ and angular momentum $I \underset{\sim}{w}$ (I = moment of inertia, $\underset{\sim}{w}$ = angular velocity). Let $\underset{\sim}{e}_{(i)}$, $i = 1,2,3$, be an orthonormal triad rigidly attached to the sphere with origin at the centre. Since

$$\frac{d\underset{\sim}{e}_{(i)}}{dt} = \underset{\sim}{w} \times \underset{\sim}{e}_{(i)} \qquad (i = 1,2,3)$$

$$\frac{d\underset{\sim}{e}_{(i)}}{dt} \cdot \frac{d\underset{\sim}{e}_{(i)}}{dt} = (\underset{\sim}{w}\times\underset{\sim}{e}_{(i)})\cdot(\underset{\sim}{w}\times\underset{\sim}{e}_{(i)}) \qquad \left(\begin{array}{l}\text{Einstein summation}\\ \text{convention used here}\end{array}\right)$$

$$= \underset{\sim}{w}\cdot(\underset{\sim}{e}_{(i)}\times(\underset{\sim}{w}\times\underset{\sim}{e}_{(i)})) = \underset{\sim}{w}\cdot(\underset{\sim}{w}(\underset{\sim}{e}_{(i)}\cdot\underset{\sim}{e}_{(i)}) - \underset{\sim}{e}_{(i)}(\underset{\sim}{e}_{(i)}\cdot\underset{\sim}{w}))$$

$$= 3\underset{\sim}{w}\cdot\underset{\sim}{w} - (\underset{\sim}{w}\cdot\underset{\sim}{e}_{(i)})(\underset{\sim}{w}\cdot\underset{\sim}{e}_{(i)})$$

$$= 3w^2 - (w_1^2 + w_2^2 + w_3^2) = 2w^2 .$$

The kinetic energy of rotation is therefore

$$\frac{1}{2} I w^2 = \frac{1}{4} I \frac{d\underset{\sim}{e}_{(i)}}{dt} \cdot \frac{d\underset{\sim}{e}_{(i)}}{dt} . \qquad \text{(A.3.1)}$$

Since

$$\frac{d\underset{\sim}{e}_{(i)}}{dt} \times \underset{\sim}{e}_{(i)} = \underset{\sim}{e}_{(i)}\times(\underset{\sim}{e}_{(i)}\times\underset{\sim}{w}) = \underset{\sim}{e}_{(i)}(\underset{\sim}{e}_{(i)}\cdot\underset{\sim}{w}) - \underset{\sim}{w}(\underset{\sim}{e}_{(i)}\cdot\underset{\sim}{e}_{(i)})$$

$$= w_i\underset{\sim}{e}_i - 3\underset{\sim}{w} = \underset{\sim}{w} - 3\underset{\sim}{w} = - 2\underset{\sim}{w} .$$

The angular momentum $I\underset{\sim}{w}$ is therefore

$$I\,\underset{\sim}{w} = \frac{1}{2}\,I\,\underset{\sim}{e}_{(i)} \times \frac{d\underset{\sim}{e}_{(i)}}{dt} \quad . \tag{A.3.2}$$

In terms of an orthonormal tetrad $e_{\alpha}^{(a)}$, $(a,\alpha = 1,\ldots,4)$ with $e_{\alpha}^{(4)}(s) = (0,0,0,-1)$, $e_{\alpha}^{(i)}(s) = (\underset{\sim}{e}_{(i)},0)$, $i = 1,2,3$ and with t replaced by proper time s , since

$$\dot{e}_{\alpha}^{(4)} \equiv de_{\alpha}^{(4)}/ds = (0,0,0,0) \;,\; \dot{e}_{\alpha}^{(i)} = (d\underset{\sim}{e}_{(i)}/dt,0) \quad i = 1,2,3$$

in (A.3.1) the rotational energy can be written as a scalar

$$\frac{1}{2}\,I\,w^2 = \frac{1}{4}\,I\,\dot{e}_{\mu}^{(a)}\,\dot{e}^{\mu}_{(a)} \quad . \tag{A.3.3}$$

Taking the above for the Lagrangian, $\underset{\sim}{L} = \frac{1}{4}\,I\,\dot{e}_{\mu}^{(a)}\,\dot{e}^{\mu}_{(a)}$. Then $\frac{\partial \underset{\sim}{L}}{\partial \dot{e}_{\nu}^{(a)}} = \frac{1}{2}\,I\,\dot{e}^{\nu}_{(a)}$ and defining spin by $S^{\mu\nu} \equiv 2\,e^{(a)[\mu}\,\frac{\partial \underset{\sim}{L}}{\partial \dot{e}^{(a)}_{\nu]}}$ gives $S^{\mu\nu} = I\,e^{(a)[\mu}\dot{e}^{\nu]}_{(a)}$. Since $\dot{e}^{4}_{(a)} = 0$, $a = 1,\ldots,4$, the components $S^{4\mu} = -S^{\mu 4}$ are zero. The spatial components of $S^{\mu\nu}$ are easily seen from (A.3.2) to be just the components of the angular momentum, e.g.,

$$S^{12} = \frac{1}{2}\,I\,e^{(a)1}\,\dot{e}^{2}_{(a)} - \frac{1}{2}\,I\,e^{(a)2}\,\dot{e}^{1}_{(a)}$$

$$= \frac{1}{2}\,I\,e_{(i)1}\,\frac{de_{(i)2}}{dt} - \frac{1}{2}\,I\,e_{(i)2}\,\frac{de_{(i)1}}{dt}$$

$$= \frac{1}{2}\,I\,(\underset{\sim}{e}_{(i)} \times \frac{d\underset{\sim}{e}_{(i)}}{dt})_3 = I\,w_3 \quad .$$

APPENDIX FOUR

Equation (3.2.2).

$$0 = g_{\sigma\tau|\rho} = g_{\sigma\tau,\rho} - \Gamma^{\alpha}_{\sigma\rho}\, g_{\alpha\tau} - \Gamma^{\alpha}_{\tau\rho}\, g_{\sigma\alpha}$$

$$0 = (g_{\sigma\tau} + dg_{\sigma\tau})|_{\rho} = (g_{\sigma\tau} + dg_{\sigma\tau})_{,\rho}$$

$$-(\Gamma^{\alpha}_{\sigma\rho} + d\Gamma^{\alpha}_{\sigma\rho})(g_{\alpha\tau} + dg_{\alpha\tau}) - (\Gamma^{\alpha}_{\tau\rho} + d\Gamma^{\alpha}_{\tau\rho})(g_{\sigma\alpha} + dg_{\sigma\alpha}) \quad .$$

Subtraction gives

$$0 = dg_{\sigma\tau,\rho} - d\Gamma^{\alpha}_{\sigma\rho}g_{\alpha\tau} - \Gamma^{\alpha}_{\sigma\rho}dg_{\alpha\tau} - d\Gamma^{\alpha}_{\tau\rho}g_{\sigma\alpha} - \Gamma^{\alpha}_{\tau\rho}dg_{\sigma\alpha}$$

i.e.,

$$0 = (dg_{\sigma\tau})|_{\rho} - d\Gamma^{\alpha}_{\sigma\rho}g_{\alpha\tau} - d\Gamma^{\alpha}_{\tau\rho}g_{\sigma\alpha} \quad .$$

Noting the similarity of this equation to the first equation (which implies $\Gamma^{\beta}_{\sigma\tau}g_{\beta\rho} = \frac{1}{2}\,(g_{\sigma\rho,\tau} + g_{\tau\rho,\sigma} - g_{\sigma\tau,\rho}))$, cyclic permutation and addition will give

$$d\Gamma^{\rho}_{\sigma\tau} = \frac{1}{2}\,(dg^{\rho}{}_{\sigma|\tau} + dg^{\rho}{}_{\tau|\sigma} - dg_{\sigma\tau}{}^{|\rho}) \quad .$$

Equation (3.2.3).

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\tau\gamma}\,\Gamma^{\tau}_{\beta\delta} - \Gamma^{\alpha}_{\tau\delta}\,\Gamma^{\tau}_{\beta\gamma} \quad .$$

In geodesic co-ordinates, $\Gamma^{\alpha}_{\beta\delta} = 0$,

$$dR^{\alpha}{}_{\beta\gamma\delta} = d\Gamma^{\alpha}_{\beta\delta,\gamma} - d\Gamma^{\alpha}_{\beta\gamma,\delta}$$

$$= 2d\Gamma^{\alpha}_{\beta[\delta,\gamma]} = 2d\Gamma^{\alpha}_{\beta[\delta|\gamma]} \ .$$

$d\Gamma^{\alpha}_{\beta\delta}$ is a tensor, and this equation holding in geodesic co-ordinates implies that it is true generally.

Equation (3.2.5).

$$B^{\tau\sigma}{}_{\rho}\,d\Gamma^{\rho}_{\sigma\tau} = B^{\tau\sigma}{}_{\rho}(dg^{\rho}{}_{(\sigma|\tau)} - \tfrac{1}{2}\,dg_{\sigma\tau}{}^{|\rho})$$

$$= (\text{div}) - \tfrac{1}{2}\,B^{\tau\sigma}{}_{\rho|\tau}dg^{\rho}{}_{\sigma} - \tfrac{1}{2}\,B^{\tau\sigma}{}_{\rho|\sigma}dg^{\rho}{}_{\tau} + \tfrac{1}{2}\,B^{\tau\sigma}{}_{\rho}{}^{|\rho}dg_{\sigma\tau}$$

$$= (\text{div}) + \tfrac{1}{2}\,(-B^{\tau\sigma\rho} - B^{\sigma\tau\rho} + B^{\sigma\rho\tau})_{|\tau}\ dg_{\rho\sigma}$$

$$= (\text{div}) + \tfrac{1}{2}\,(-B^{\tau(\sigma\rho)} - B^{(\sigma\tau\rho)} + B^{(\sigma\rho)\tau})_{|\tau}\ dg_{\rho\sigma}$$

(since $dg_{\rho\sigma}$ is symmetric). Therefore,

$$B^{\tau\sigma}{}_{\rho}\,d\Gamma^{\rho}_{\sigma\tau} = (\text{div}) + \tfrac{1}{4}\,(-B^{\tau\sigma\rho} - B^{\tau\rho\sigma} - B^{\sigma\tau\rho} - B^{\rho\tau\sigma} + B^{\sigma\rho\tau} + B^{\rho\sigma\tau})_{|\tau}\ dg_{\rho\sigma}$$

$$= (\text{div}) + \tfrac{1}{2}\,(B^{\rho[\sigma\tau]} + B^{\sigma[\rho\tau]} - B^{\tau(\rho\sigma)})_{|\tau}dg_{\rho\sigma} \quad .$$

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